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A Constraint-Stabilized Time-Stepping Approach for Rigid Multibody Dynamics with Joints, Contact and Friction

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SUMMARY

We present a method for achieving geometrical constraint stabilization for a linear-complementarity-based time-stepping scheme for rigid multibody dynamics with joints, contact, and friction. The method requires the solution of only one linear complementarity problem per step. We prove that the velocity stays bounded and that the constraint infeasibility is uniformly bounded in terms of the size of the time step and the current value of the velocity. Several examples, including one for joint-only systems, are used to demonstrate the constraint stabilization effect. **Subject Index** 65L80, 90C33, 70E55, 74M10, 74M15

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1. Introduction

Simulating the dynamics of a system with several rigid bodies and with joint, contact (noninterpenetration), and friction constraints is an important part of virtual reality and robotics simulations.

If the simulation has only joint constraints, then the problem is a differential algebraic equation (DAE) [21, 8]. However, the nonsmooth nature of the noninterpenetration and friction constraints requires the use of specialized techniques. Approaches used in the past for simulating rigid multibody dynamics with contact and friction include piecewise DAE approaches [21], acceleration-force linear complementarity problem (LCP) approaches [11, 17], penalty approaches [16, 28], and velocity-impulse LCP-based time-stepping methods [29, 30, 3, 4].

In this work we use the last approach, which has the advantage that it does not suffer from the lack of existence of a solution that could appear in the first two approaches [11, 30]. It also does not suffer from the artificial stiffness that is introduced by the third approach.

On the other hand, some of the features that are well studied and well understood for numerical schemes for DAE, such as accommodation of stiff forces or constraint stabilization, cannot be readily extended to the velocity-impulse LCP approach. For example, accommodating stiff forces by implicit methods may require the resolution of a nonlinear complementarity problem whose solution set is likely nonconvex in some cases, since it is nonconvex for linear complementarity formulations [1]. This situation can be remedied by using a linearized backward Euler approach, which results in an unconditionally consistent linear complementarity problem for the case where the stiffness is generated by springs and dampers [2].

In this work we discuss the problem of achieving geometrical (noninterpenetration and joint) constraint stabilization for time-stepping methods for rigid multibody dynamics with contact, joints, and friction. The problem has been tackled by using nonlinear complementarity problems [29], an LCP followed by a nonlinear projection approach that includes nonlinear inequality constraints [2], and a postprocessing method [12] that uses one potentially nonconvex LCP based on the stiff method developed in [2] followed by one convex LCP for constraint stabilization. When applied to joint-only systems, the method from [12] belongs to the set of postprocessing methods defined in [7, 9].

In this work we show that geometrical constraint stabilization for rigid multibody dynamics simulations with joints, contact, and friction can be achieved while solving only one LCP per step, of comparable complexity to the first LCP in [12] and to the LCP in [3, 2]. This is done by choosing a free term (right-hand side) of the LCP that depends on the geometrical constraint infeasibility. Therefore, the infeasibility will affect the energy balance of the time-stepping scheme. The main challenge is to prove that, under certain assumptions, the effect of the infeasibility over the solution of the LCP goes to 0 as the time step goes to 0, uniformly over the entire time-simulation interval. We address this challenge in Section 4.

To our knowledge, our work is novel in several respects.

- We provide an analysis of a constraint stabilization mechanism where the subproblem is an LCP. All other approaches involve either additional nonlinear projections or apply only to DAE, where the main subproblems are either linear or nonlinear systems of equations.

- We show that a rigid multibody simulation framework can be defined in such a fashion that the simulation can progress with a constant time step and remain stable for sufficiently small time step, while solving only one LCP per time step, even when totally plastic collisions occur.

  This is very different from the integrate-detect-restart strategy that is the basis of most current methods of rigid multibody dynamics with contact and friction simulations [14, 20, 11, 3, 29]. Such approaches solve a subproblem of a complexity comparable to the one that is solved here, based on an initial approximation of the active set, and then attempt to advance the simulation by one time step. If one event (collision or contact takeoff) is detected, then the simulation is backtracked up to the event, the active set is updated, and the simulation is restarted. The problem is that the number of linear complementarity problems to be solved per unit of simulation time is unpredictable, since there is no apriori bound to the number of collisions that can occur per unit of simulation time. This problem is related to the one that appears in simulating bouncing balls with this approach [19], which we do not address here because we consider only totally plastic collisions.

  Since in the absence of contacts and friction our approach reduces to an Euler method, the method presented here cannot exceed order 1. As long as we do not detect the events and we do not backtrack, we cannot expect to extend the method to orders higher than one even if we change the numerical scheme. Nevertheless, we believe that such an approach is valuable from a practical perspective, when many collisions are expected over a small time interval (as happens on a pool table, for example) for which high-order integration may not make sense because of the extreme sensitivity of the outcome to the many parameters involved. Stabilized, low-order methods that solve a fixed number of subproblems per step like our approach may be very useful when a human is in the loop (interactive simulation), where the main goal is to provide an approximation of the dynamics that is acceptable, within the tolerance of human perception, with a low amount of computational effort.
- When our approach is reduced to the DAE case, which appears when only joint constraints are considered, we obtain an order 1 method that achieves constraint stabilization while solving only one linear system per step and that does not depend on any parameter tuning. This may also be a novel result. For example, applying the postprocessing method [7] to DAE in our setup would require the solution of two linear systems per step.

  The postprocessing method can be implemented to use only one matrix per step. When a direct method is used to solve the linear systems that appear in our method and the postprocessing method then only one matrix factorization per step is needed by either method and the computational efforts are comparable. If an iterative method is used, however, then the postprocessing method may need, at least in the worst case, twice as much computational effort as our method. Moreover, the reduction of computational effort, that appears in the direct solver case for DAE, does not extend to the postprocessing method described in [12] when inequality (contact and friction) constraints are considered, since an inequality-constrained quadratic program cannot (in general) be re-solved at a low computational cost if its right-hand side is changed.

When this paper was in the final stages of completion, we became aware of a method based on [3] that uses a constraint stabilization approach that is similar to the one discussed here but that does depend on an additional parameter. The paper [13] contains no proof of stabilization, although it contains extensive numerical validation for realistic robot grasp stability applications.

In this work we do not address the issue of convergence as the time step goes to 0, which for previous linear-complementarity-based schemes [29, 3] was proved in a differential inclusion sense [30]. Existence of classical solutions for the continuous time problem does not hold in general [11, 30].

The paper is organized as follows. In Section 2 we introduce the time-stepping scheme based on the LCP, and we discuss relevant properties of strictly convex quadratic programs and their connection to the LCP. In Section 3 we prove a stability property of the solution of certain quadratic programs with respect to their right-hand side. In Section 4 we prove our main stabilization result. In Section 5 we discuss the behavior of the method when applied to a problem with joint-only constraints. In Section 6 we present numerical validation of the concepts described and proved in the paper. In the Appendix we prove several results concerning the behavior of recursive inequalities that are relevant to the proofs in Section 4.

2. The Linear Complementarity Subproblem of the Time-Stepping Scheme

In this section, we review a velocity-impulse LCP-based time-stepping scheme that uses an Euler discretization [3, 29]. In the following, q and v constitute, respectively, the generalized position and generalized velocity vector of a system of several bodies [21].

2.1. Model Constraints

Throughout this subsection we make use of **complementarity notation**. If $a, b \in I\!R$, we say that a is complementary to b, and we denote it by $a \perp b$ or $a \geq 0 \perp b \geq 0$ if $a \geq 0$, $b \geq 0$, and $ab = 0$.

2.1.1. Geometrical Constraints Joint constraints (2.1) and noninterpenetration constraints (2.3) involve only the position variable and depend on the shape of the bodies and the type of constraints involved. We call them geometrical constraints.

Joint Constraints. Joint constraints are described by the equations

$$\Theta^{(i)}(q) = 0,\ i = 1, 2, \ldots, m\,. \tag{2.1}$$

Here, $\Theta^{(i)}(q)$ are sufficiently smooth functions. We denote by $\nu^{(i)}(q)$ the gradient of the corresponding function, or

$$\nu^{(i)}(q) = \nabla_q \Theta^{(i)}(q), \quad i = 1, 2, \ldots, m. \tag{2.2}$$

The impulse exerted by a joint on the system is $c_\nu^{(i)} \nu^{(i)}(q)$, where $c_\nu^{(i)}$ is a scalar related to the Lagrange multiplier of classical constrained dynamics [21].

Noninterpenetration Constraints. Noninterpenetration constraints are defined in terms of a continuous signed distance function between the two bodies $\Phi^{(j)}(q)$ [5]. The noninterpenetration constraints become

$$\Phi^{(j)}(q) \geq 0, \quad j = 1, 2, \ldots, p. \tag{2.3}$$

The function $\Phi^{(j)}(q)$ is generally not differentiable everywhere. In Section 4 we discuss sufficient conditions for local differentiability of $\Phi^{(j)}(q)$. In the following, we may refer to j as the *contact* j, although the contact is truly active only when $\Phi^{(j)}(q) = 0$. We denote the normal at contact (j) by

$$n^{(j)}(q) = \nabla_q \Phi^{(j)}(q), \quad j = 1, 2, \ldots, p. \tag{2.4}$$

When the contact is active, it can exert a compressive normal impulse, $c_n^{(j)} n^{(j)}(q)$, on the system, which is quantified by requiring $c_n^{(j)} \geq 0$. The fact that the contact must be active before a nonzero compression impulse can act is expressed by the complementarity constraint

$$\Phi^{(j)}(q) \geq 0 \perp c_n^{(j)} \geq 0, \quad j = 1, 2, \ldots, p. \tag{2.5}$$

2.1.2. Frictional Constraints Frictional constraints are expressed by means of a discretization of the Coulomb friction cone [2, 3, 29]. For a contact $j \in \{1, 2, \ldots, p\}$, we take a collection of coplanar vectors $d_i^{(j)}(q)$, $i = 1, 2, \ldots, m_C^{(j)}$, which span the plane tangent at the contact (though the plane may cease to be tangent to the contact normal when mapped in generalized coordinates [5]). The convex cover of the vectors $d_i^{(j)}(q)$ should approximate the transversal shape of the friction cone. In two-dimensional mechanics, the tangent plane is one dimensional, its transversal shape is a segment, and only two such vectors $d_1^{(j)}(q)$ and $d_2^{(j)}(q)$ are needed in this formulation. We denote by $D^{(j)}(q)$ a matrix whose columns are $d_i^{(j)}(q) \neq 0$, $i = 1, 2, \ldots, m_C^{(j)}$, that is, $D^{(j)}(q) = \left[d_1^{(j)}(q), d_2^{(j)}(q), \ldots, d_{m_C^{(j)}}^{(j)}(q) \right]$. A tangential impulse is $\sum_{i=1}^{m_C^{(j)}} \beta_i^{(j)} d_i^{(j)}(q)$, where $\beta_i^{(j)} \geq 0$, $i = 1, 2, \ldots, m_C^{(j)}$. We assume that the tangential contact description is balanced, that is,

$$\forall 1 \leq i \leq m_C^{(j)},\ \exists k,\ 1 \leq k \leq m_C^{(j)} \text{ such that } d_i^{(j)}(q) = -d_k^{(j)}(q). \tag{2.6}$$

The friction model ensures maximum dissipation for given normal impulse $c_n^{(j)}$ and velocity v and guarantees that the total contact force is inside the discretized cone. We express this model as

$$\begin{array}{rlcl} D^{(j)^T}(q)v + \lambda^{(j)} e^{(j)} & \geq 0 & \perp & \beta^{(j)} \geq 0, \\ \mu c_n^{(j)} - e^{(j)^T} \beta^{(j)} & \geq 0 & \perp & \lambda^{(j)} \geq 0. \end{array} \tag{2.7}$$

Here $e^{(j)}$ is a vector of ones of dimension $m_C^{(j)}$, $e^{(j)} = (1, 1, \ldots, 1)^T$, $\mu^{(j)} \geq 0$ is the Coulomb friction parameter, and $\beta^{(j)}$ is the vector of tangential impulses $\beta^{(j)} = \left(\beta_1^{(j)}, \beta_2^{(j)}, \ldots, \beta_{m_C^{(j)}}^{(j)} \right)^T$. The additional variable $\lambda^{(j)} \geq 0$ is approximately equal to the norm of the tangential velocity at the contact, if there is relative motion at the contact, or $\left|\left| D(q)^{(j)^T} v \right|\right| \neq 0$ [3, 29].

Notation. We denote by $M(q)$ the symmetric, positive definite mass matrix of the system in the generalized coordinates q and by $k(t, q, v)$ the external force. All quantities described in this section associated with contact j are denoted by the superscript $^{(j)}$. When we use a vector or matrix norm whose index is not specified, it is the 2 norm.

2.2. The Linear Complementarity Problem

Let $h_l > 0$ be the time step at time $t^{(l)}$, when the system is at position $q^{(l)}$ and velocity $v^{(l)}$. We have that $h_l = t^{(l+1)} - t^{(l)}$. We choose the new position to be $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$, where $v^{(l+1)}$ is determined by enforcing the simulation constraints.

The geometrical constraints are enforced at the velocity level by linearization. For joint constraints the linearization leads to

$$\Theta^{(i)}(q^{(l)}) + h_l \nabla_q \Theta^{(i)^T}(q^{(l)}) v^{(l+1)} = \Theta^{(i)}(q^{(l)}) + h_l \nu^{(i)^T}(q^{(l)}) v^{(l+1)} = 0, \quad i = 1, 2, \ldots, m. \tag{2.8}$$

For a noninterpenetration constraint of index j, $\Phi^{(j)}(q) \geq 0$, linearization at $q^{(l)}$ for one time step amounts to $\Phi^{(j)}(q^{(l)}) + h_l \nabla_q \Phi^{(j)^T}(q^{(l)}) v^{(l+1)} \geq 0$, that is, after including the complementarity constraints (2.5),

$$\nabla_q \Phi^{(j)^T}(q^{(l)}) v^{(l+1)} + \frac{\Phi^{(j)}(q^{(l)})}{h_l} \geq 0 \perp c_n^{(j)} \geq 0. \tag{2.9}$$

For computational efficiency, only the contacts that are imminently active are included in the dynamical resolution and linearized, and their set is denoted by $\mathcal{A}$. One practical way of determining $\mathcal{A}$ is by including all j for which $\Phi^{(j)}(q) \leq \hat{\epsilon}$, where $\hat{\epsilon}$ is a sufficiently small quantity.

If a collision occurs, then a collision resolution, possibly with energy restitution, needs to be applied [3]. In our setup a collision occurs at step l for a contact j if the first inequality in (2.9) is satisfied with equality, and at step $l-1$ it was satisfied as a strict inequality.

In this work we assume that no energy lost during collision is restituted; hence we avoid the need to consider a compression LCP followed by decompression LCP [3]. The relation (2.9) is sufficient to accommodate totally plastic collisions.

To completely define the LCP subproblem, we use an Euler discretization of Newton's law, which results in the following equation:

$$M(q^{(l)}) \left(v^{(l+1)} - v^{(l)} \right) = h_l k \left(t^{(l)}, q^{(l)}, v^{(l)} \right) + \sum_{i=1}^{m} c_\nu^{(i)} \nu^{(i)}(q^{(l)}) + \sum_{j \in \mathcal{A}} \left(c_n^{(j)} n^{(j)}(q^{(l)}) + \sum_{i=1}^{m_C^{(j)}} \beta_i^{(j)} d_i^{(j)}(q^{(l)}) \right).$$

After collecting all the constraints introduced above, with the geometrical constraints replaced by their linearized versions (2.8) and (2.9), we obtain the following mixed LCP:

$$\begin{bmatrix} M^{(l)} & -\widetilde{\nu} & -\widetilde{n} & -\widetilde{D} & 0 \\ \widetilde{\nu}^T & 0 & 0 & 0 & 0 \\ \widetilde{n}^T & 0 & 0 & 0 & 0 \\ \widetilde{D}^T & 0 & 0 & 0 & \widetilde{E} \\ 0 & 0 & \widetilde{\mu} & -\widetilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ c_\nu \\ c_n \\ \widetilde{\beta} \\ \lambda \end{bmatrix} + \begin{bmatrix} -Mv^{(l)} - h_l k^{(l)} \\ \Upsilon \\ \Delta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \rho \\ \widetilde{\sigma} \\ \zeta \end{bmatrix} \tag{2.10}$$

$$\begin{bmatrix} c_n \\ \widetilde{\beta} \\ \lambda \end{bmatrix}^T \begin{bmatrix} \rho \\ \widetilde{\sigma} \\ \zeta \end{bmatrix} = 0, \qquad \begin{bmatrix} c_n \\ \widetilde{\beta} \\ \lambda \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \rho \\ \widetilde{\sigma} \\ \zeta \end{bmatrix} \geq 0\,. \tag{2.11}$$

Here $\widetilde{\nu} = [\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(m)}]$, $c_\nu = [c_\nu^{(1)}, c_\nu^{(2)}, \ldots, c_\nu^{(m)}]^T$, $\widetilde{n} = [n^{(j_1)}, n^{(j_1)}, \ldots, n^{(j_s)}]$, $c_n = [c_n^{(j_1)}, c_n^{(j_2)}, \ldots, c_n^{(j_s)}]^T$, $\widetilde{\beta} = [\beta^{(j_1)T}, \beta^{(j_2)T}, \ldots, \beta^{(j_s)T}]^T$, $\widetilde{D} = [D^{(j_1)}, D^{(j_2)}, \ldots, D^{(j_s)}]$, $\lambda = [\lambda^{(j_1)}, \lambda^{(j_2)}, \ldots, \lambda^{(j_s)}]^T$, $\widetilde{\mu} = diag(\mu^{(j_1)}, \mu^{(j_2)}, \ldots, \mu^{(j_s)})^T$, $\Upsilon = \frac{1}{h}\left(\Theta^{(1)}, \Theta^{(2)}, \ldots, \Theta^{(m)}\right)^T$, $\Delta = \frac{1}{h}\left(\Phi^{(j_1)}, \Phi^{(j_2)}, \ldots, \Phi^{(j_s)}\right)^T$ and

$$\widetilde{E} = \begin{bmatrix} e^{(j_1)} & 0 & 0 & \cdots & 0 \\ 0 & e^{(j_2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{(j_s)} \end{bmatrix}$$

are the lumped LCP data, and $\mathcal{A} = \{j_1, j_2, \ldots, j_s\}$ are the active contact constraints. The vector inequalities in (2.11) are to be understood componentwise. We use the ˜ notation to indicate that the quantity is obtained by properly adjoining blocks that are relevant to the aggregate joint or contact constraints. The problem is called mixed LCP because it contains both equality and complementarity constraints.

To simplify the presentation, we have not explicitly included the dependence of the parameters in (2.10–2.11) on $q^{(l)}$. Also, $M^{(l)} = M(q^{(l)})$ is the value of the mass matrix at time $t^{(l)}$, and $k^{(l)} = k(t^{(l)}, q^{(l)}, v^{(l)})$ represents the external force at time $t^{(l)}$. We denote $\widehat{k}^{(l)}$ as follows:

$$\widehat{k}^{(l)} = -Mv^{(l)} - h_l k^{(l)}.$$

We note that a similar method, also based on the algorithm in [3], has been used in [13]. Instead of Δ and Υ in (2.10), however, that method uses the quantities

$$\widehat{\Delta} = \gamma\Delta, \text{ and } \widehat{\Upsilon} = \gamma\Upsilon \tag{2.12}$$

where $0 < \gamma < 1$ is a parameter.

2.3. A convex quadratic program that is locally equivalent to the LCP (2.10–2.11)

The key observation in obtaining the geometrical constraint stabilization results is that a velocity solution of (2.10)–(2.11) is also the solution of the strictly convex quadratic program (2.13) whose right-hand side depends on that particular solution of (2.10)–(2.11) [1]. This quadratic program (2.13) is only locally equivalent to (2.10–2.11), namely, at that particular solution of (2.13).

Although we do not use this quadratic program (2.13) to determine the velocity (and in effect we cannot, since its right-hand side depends on the unknown velocity), it is a useful tool for proving our results. For completeness, we include a proof of the local equivalence, which is different from the one in [1].

Theorem 2.1. *Consider a solution* $\left(v^{(l+1)}, c_\nu, c_n, \tilde{\beta}, \lambda\right)$ *of (2.10–2.11). Define* $\Gamma = \tilde{\mu}\lambda$. *Then* $v^{(l+1)}$ *is a solution of the quadratic program*

$$\begin{array}{llcll} \min_v & \frac{1}{2}v^T M^{(l)} v + \widehat{k}^{(l)^T} v & & & \\ \text{subject to} & n^{(j)^T} v + \mu^{(j)} d_i^{(j)^T} v & \geq & -\left(\Gamma^{(j)} + \Delta^{(j)}\right), & j \in \mathcal{A},\ i = 1, 2, \ldots, m_C^{(j)} \\ & \nu^{(i)^T} v & = & -\Upsilon^{(i)}, & i = 1, 2, \ldots, m. \end{array} \tag{2.13}$$

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Note An alternative way of representing the quadratic program (2.13) is by aggregating the data corresponding to one contact j in a block to obtain the following quadratic program:

$$\begin{array}{rrcll} \min_v & \frac{1}{2} v^T M^{(l)} v + \widehat{k}^{(l)^T} v & & & \\ \text{subject to} & e^{(j)} n^{(j)^T} v + \mu^{(j)} D^{(j)^T} v & \geq & -\left(\Gamma^{(j)} + \Delta^{(j)}\right) e^{(j)}, & j \in \mathcal{A} \\ & \nu^{(i)^T} v & = & -\Upsilon^{(i)}, & i = 1, 2, \ldots, m. \end{array} \tag{2.14}$$

Proof To simplify the presentation, we do not include in the proof the index l of the current time step. Within this proof, we therefore denote $v^{(l)}$ by v and $M^{(l)}$ by M.

We first show that from any solution $\left(v^{(l+1)}, c_\nu, c_n, \tilde{\beta}, \lambda\right)$ of (2.10–2.11), we can construct another solution $\left(v^{(l+1)}, c_\nu, c_n, \tilde{\beta^*}, \lambda\right)$ of (2.10)–(2.11) that satisfies, for any $j \in \mathcal{A}$,

$$\mu^{(j)} c_n^{(j)} = \sum_{i=1}^{m_C^{(j)}} \beta_i^{*(j)}. \tag{2.15}$$

We fix one active contact constraint index $j^* \in \mathcal{A}$. The part of (2.10–2.11) that is influenced by the tangential impulse at contact (j^*), $\beta^{(j^*)}$ consists of the equations

$$Mv - \sum_{i=1}^{m} c_\nu^{(i)} \nu^{(i)} - \sum_{j \in \mathcal{A}} c_n^{(j)} n^{(j)} - \sum_{j \in \mathcal{A}} \sum_{i=1}^{m_C^{(j)}} \beta_i^{(j)} d_i^{(j)} = -\widehat{k} \tag{2.16}$$

$$\lambda^{(j^*)} + d_i^{T^{(j^*)}} v \geq 0 \quad \perp \quad \beta_i^{(j^*)} \geq 0, \; i = 1, 2, \ldots, m_C^{(j^*)} \tag{2.17}$$

$$\mu^{(j^*)} c_n^{(j^*)} - \sum_{i=1}^{m_C^{(j^*)}} \beta_i^{(j^*)} \geq 0 \quad \perp \quad \lambda^{(j^*)} \geq 0. \tag{2.18}$$

We have two cases. If $\lambda^{(j^*)} > 0$, then from (2.18) we have that

$$\mu^{(j^*)} c_n^{(j^*)} = \sum_{i=1}^{m_C^{(j^*)}} \beta_i^{(j^*)},$$

and (2.15) is satisfied by simply choosing $\beta^{*(j^*)} = \beta^{(j^*)}$ and not changing anything. Assume now that $\lambda^{(j^*)} = 0$ and that

$$\zeta^{(j^*)} = \mu^{(j^*)} c_n^{(j^*)} - \sum_{i=1}^{m_C^{(j^*)}} \beta_i^{(j^*)} > 0$$

(since the case $\zeta^{(j*)} = 0$ reduces to the one already analyzed). We define

$$\beta_i^{*(j^*)} = \beta_i^{(j^*)} + \frac{\zeta^{(j^*)}}{m_C^{(j^*)}} > 0, \; i = 1, 2, \ldots, m_C^{(j^*)},$$

which clearly leads to (2.15) being satisfied.

Since $\lambda^{(j^*)} = 0$, we have that (2.18) is also satisfied when $\beta^{(j^*)}$ is replaced by $\beta^{*(j^*)}$. Also, since the friction cone approximation is balanced, we obtain from (2.6) and (2.17) that

$d_i^{(j^*)^T} v \geq 0$ and that $-d_i^{(j^*)^T} v \geq 0$ for any $i = 1, 2, \ldots, m_C^{(j^*)}$, which implies that $d_i^{(j^*)^T} v = 0$ for $i = 1, 2, \ldots, m_C^{(j^*)}$. Therefore $\beta_i^{*(j^*)}$ satisfies (2.17) as well for $i = 1, 2, \ldots, m_C^{(j^*)}$. Finally, from (2.6) we obtain that

$$\sum_{i=1}^{m_C^{(j^*)}} d_i^{(j^*)} = 0,$$

which implies, from the definition of $\beta_i^{*(j^*)}$ that

$$\sum_{i=1}^{m_C^{(j^*)}} \beta_i^{(j^*)} d_i^{(j)} = \sum_{i=1}^{m_C^{(j^*)}} \beta_i^{(j^*)} d_i^{(j)} + \frac{\zeta^{(j^*)}}{m_C^{(j^*)}} \sum_{i=1}^{m_C^{(j^*)}} d_i^{(j^*)} = \sum_{i=1}^{m_C^{(j^*)}} \beta_i^{(*j^*)} d_i^{(j)}.$$

Using the last equation in (2.16), we obtain that (2.16) continues to be satisfied, after we replace $\beta_i^{(j^*)}$ by $\beta_i^{*(j^*)}$.

Using this argument for all $j^* \in \mathcal{A}$, we obtain a solution of (2.10)–(2.11), $\left(v^{(l+1)}, c_\nu, c_n, \tilde{\beta}^*, \lambda\right)$, that satisfies (2.15). To simplify the notation, we assume that (2.15) is satisfied by the given solution in our assumption, that is, $\tilde{\beta} = \tilde{\beta}^*$, and we proceed to prove the rest of the claim.

For a contact $j \in \mathcal{A}$, the constraints in (2.10–2.11) are

$$n^{(j)^T} v - \mu^{(j)} \lambda^{(j)} + \Gamma^{(j)} + \Delta^{(j)} \geq 0 \quad \perp \quad c_n^{(j)} \geq 0 \tag{2.19}$$

$$\lambda^{(j)} + d_i^{(j)^T} v \geq 0 \quad \perp \quad \beta_i^{(j)} \geq 0, \; i = 1, 2, \ldots, m_C^{(j)} \tag{2.20}$$

$$\mu^{(j)} c_n^{(j)} - \sum_{i=1}^{m_C^{(j)}} \beta_i^{(j)} \geq 0 \quad \perp \quad \lambda^{(j)} \geq 0, \tag{2.21}$$

where we used the definition in the assumption that $\Gamma^{(j)} = \mu^{(j)} \lambda^{(j)}$. We claim that (2.19–2.21) imply that

$$n^{(j)^T} v + \mu^{(j)} d_i^{(j)^T} v + \Gamma^{(j)} + \Delta^{(j)} \geq 0 \quad \perp \quad \beta_i^{(j)} \geq 0, \; i = 1, 2, \ldots, m_C^{(j)}. \tag{2.22}$$

We have two cases to consider. If $c_n^{(j)} = 0$, then from (2.20) and (2.21) we must have that $\beta_i^{(j)} = 0$ for $i = 1, 2, \ldots, m_C^{(j)}$. Multiplying the left side of the equation (2.20) by $\mu^{(j)}$ and adding to it the equation (2.19) for $i = 1, 2, \ldots, m_C^{(j)}$, we obtain that (2.22) holds.

If $c_n^{(j)} > 0$, then we obtain from (2.19) that

$$n^{(j)^T} v - \mu^{(j)} \lambda^{(j)} + \Gamma^{(j)} + \Delta^{(j)} = 0.$$

Again, multiplying (2.20) by $\mu^{(j)}$ and adding to it the previous relation, we obtain (2.22).

So (2.22) must hold in any case. To prove our theorem in full generality, we now need to separate the active set in two subsets:

$$\mathcal{A}_1 = \left\{ j \in \mathcal{A} | \mu^{(j)} > 0 \right\}, \quad \mathcal{A}_2 = \left\{ j \in \mathcal{A} | \mu^{(j)} = 0 \right\}.$$

Whenever $j \in \mathcal{A}_1$, and using (2.15) with $\tilde{\beta} = \tilde{\beta}^*$, we can rewrite the reaction impulse in (2.10)–(2.11) corresponding to the contact (j) as

$$c_n^{(j)} n^{(j)} + \sum_{i=1}^{m_C^{(j)}} \beta_i^{(j)} d_i^{(j)} = \sum_{i=1}^{m_C^{(j)}} \frac{\beta_i^{(j)}}{\mu^{(j)}} \left(n^{(j)} + \mu^{(j)} d_i^{(j)} \right).$$

Using now (2.10–2.11), (2.19) for $j \in \mathcal{A}_2$ and (2.22) for $j \in \mathcal{A}_1$, where we divide the right-hand part of the relation by $\mu^{(j)} > 0$, we obtain that

$$\begin{array}{rcl} Mv - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j \in \mathcal{A}_1} \sum_{i=1}^{m_C^{(j)}} \frac{\beta_i^{(j)}}{\mu^{(j)}} \left(n^{(j)} + \mu^{(j)} d_i^{(j)} \right) - \sum_{j \in \mathcal{A}_2} c_n^{(j)} n^{(j)} & = & -\widehat{k}^{(l)} \\ n^{(j)^T} v + \mu^{(j)} d_i^{(j)^T} v + \Gamma^{(j)} + \Delta^{(j)} \geq 0 \perp \frac{\beta_i^{(j)}}{\mu^{(j)}} \geq 0, \; i = 1, 2, \ldots, m_C^{(j)}, & & j \in \mathcal{A}_1. \\ n^{(j)^T} v + \Gamma^{(j)} + \Delta^{(j)} \geq 0 \perp c_n^{(j)} \geq 0, & & j \in \mathcal{A}_2 \\ \nu^{(i)^T} v = -\Upsilon^{(i)}, \; i = 1, 2, \ldots, p. & & \end{array}$$

An inspection of the last relation shows that it contains optimality conditions for (2.13), with the third constraint repeated $m_C^{(j)}$ times when $j \in \mathcal{A}_2$. This shows that v is a solution of the strictly convex quadratic program (2.13) as claimed. The proof is complete. $\diamond$

For a solution of (2.10)–(2.11) there may be some freedom in the choice of the multipliers λ, at the contacts where there is no slip velocity. However, there is always a minimal choice of λ, as we now show.

Lemma 2.2. *Consider a solution* $\left(v^{(l+1)}, c_\nu, c_n, \tilde{\beta}, \lambda\right)$ *of (2.10)–(2.11). Then there exists a choice* λ^* *such that* $\left(v^{(l+1)}, c_\nu, c_n, \tilde{\beta}, \lambda^*\right)$ *is also a solution of (2.10)–(2.11) such that, for any* $j \in \mathcal{A}$*, we have that equality is attained in*

$$D^{(j)^T} v^{(l+1)} + \lambda^{*(j)} e^{(j)} \geq 0$$

for at least one entry.

Proof Let $j \in \mathcal{A}$ be the index of an active contact. Then the only relations affected by $\lambda^{(j)}$ are the ones describing the friction model, (2.7):

$$\begin{array}{rcl} D^{(j)^T} v^{(l+1)} + \lambda^{(j)} e^{(j)} & \geq 0 \quad \perp & \beta^{(j)} \geq 0, \\ \mu^{(j)} c_n^{(j)} - e^{(j)^T} \beta^{(j)} & \geq 0 \quad \perp & \lambda^{(j)} \geq 0. \end{array} \tag{2.23}$$

If equality is attained at least for one entry in the first inequality, then we can simply define $\lambda^{*(j)} = \lambda^{(j)}$. If equality is not attained, then $\lambda^{(j)} > 0$, and we must have, by the complementarity relation, that $\beta^{(j)} = 0$ and, from the second complementarity relation, that $\mu^{(j)} c_n^{(j)} = 0$. Choose now

$$\lambda^{*(j)} = - \min_{i=1,2,\ldots,m_C^{(j)}} \left\{ d_i^{(j)^T} v^{(l+1)} \right\}, \quad j \in \mathcal{A}.$$

We have that $\lambda^{*(j)} \geq 0$, from our assumption (2.6) of a balanced approximation to the friction cone. It can now be immediately seen that if we replace $\lambda^{(j)}$ with $\lambda^{*(j)}$, then the relations (2.23) still hold. In addition, at least one entry in the first equation of (2.23) with $\lambda^{(j)}$ replaced by $\lambda^{*(j)}$ will be satisfied with equality (the one for which the minimum in the displayed equation is attained). The proof is complete. ⋄

2.4. Constraint Qualification and the Pointed Friction Cone Assumption

To approach geometrical constraint stabilization by our method, we need to develop several results quantifying the dependence of the solution of strictly convex quadratic programs with respect to the free term of the constraints, and in particular of the solution v of (2.13) with respect to Δ and Υ. The main tool in obtaining these results will be the pointed friction cone assumption, which we show that it is equivalent to the Mangasarian-Fromovitz constraint qualification for (2.13).

2.4.1. Mangasarian-Fromovitz Constraint Qualification for Quadratic Programs Consider the quadratic program

$$\begin{array}{rrcl} \text{minimize} & q^T x + \frac{1}{2} x^T Q x & & \\ \text{subject to} & A^T x + \alpha & \geq & 0 \\ & B^T x + \beta & = & 0, \end{array} \tag{2.24}$$

where $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times p}$, $\alpha \in \mathbb{R}^m$, and $\beta \in \mathbb{R}^p$.

We say that MFCQ holds for (2.24) at any point $x \in \mathbb{R}^n$ if the following hold:

$$\text{(MFCQ)} \qquad \begin{array}{l} 1.\ B \text{ has full row rank.} \\ 2.\ \exists f \in \mathbb{R}^m,\ f < 0, \text{ and } d \in \mathbb{R}^n, \quad \text{such that} \quad A^T d = f,\ B^T d = 0. \end{array}$$

In the context of nonlinear programming, MFCQ is defined at a point x and involves only the inequalities active at x [22, 23]. MFCQ is the essential condition for nonlinear programs to behave well with respect to perturbations [27], a fact that will also be exploited here.

In addition, MFCQ is important because it allows us to work with a constraint qualification that is weaker than linear independence of the constraints of (2.24). When applying these concepts to (2.13) we have shown that the linear independence constraint qualification is too strong of a concept for three-dimensional configurations [1].

A useful characterization of MFCQ is obtained by the use of duality. We obtain that [1]

$$\text{MFCQ holds} \;\Leftrightarrow\; \left.\begin{array}{l} A\mu + B\nu = 0 \\ \mu \in \mathbb{R}^m,\ \mu \geq 0,\ \nu \in \mathbb{R}^p \end{array}\right\} \Rightarrow \mu = 0,\ \nu = 0. \tag{2.25}$$

An important consequence of (2.24) satisfying MFCQ is the following.

Lemma 2.3. *If the quadratic program (2.24) satisfies MFCQ, then it is feasible for any $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^p$. If, in addition, the matrix Q is positive definite, then the quadratic program has a solution.*

Proof The first part of the claim follows from [27]. The second part of the claim is obvious from the strict convexity of the objective function of (2.24). ⋄

2.4.2. Pointed Friction Cone We now define a friction cone regularity assumption. We define the friction cone to be the portion in the velocity space that can be covered by feasible constraint interaction impulses, or

$$FC(q) = \left\{ t = \tilde{\nu} c_\nu + \tilde{n} c_n + \tilde{D}\tilde{\beta} \,\middle|\, c_n \geq 0,\ \tilde{\beta} \geq 0,\ \left|\left|\beta^{(j)}\right|\right|_1 \leq \mu^{(j)} c_n^{(j)},\ \forall j \in \mathcal{A} \right\}. \tag{2.26}$$

Clearly, the cone $FC(q)$ is a convex set.

Definition [30]: We say that the friction cone $FC(q)$ is pointed if it does not contain any proper linear subspace.

This assumption is essential in ensuring that the limits of the solutions of the time-stepping scheme (2.10)–(2.11) converge to a weak solution of the continuous problem [30]. By using the duality theory, we have the following result [1]:

$$\text{FC(q) is pointed} \Leftrightarrow \forall \left(c_\nu, c_n \geq 0, \tilde{\beta} \geq 0 \right) \neq 0 \text{ such that } \left|\left|\beta^{(j)}\right|\right|_1 \leq \mu^{(j)} c_n^{(j)},\ \forall j \in \mathcal{A}$$
$$\text{we must have that } \tilde{\nu} c_\nu + \tilde{n} c_n + \tilde{D}\tilde{\beta} \neq 0. \tag{2.27}$$

The pointed friction cone assumption plays an important part in the study of the limit case $h \to 0$ [30]. In our case, we have the following important result.

Theorem 2.4. *Assume that the friction cone $FC(q)$ is pointed. Then the mixed LCP (2.10)–(2.11) has a solution.*

Proof It is an immediate consequence of the results in [26]. $\diamond$

It is also of interest to write an alternative description of the pointedness of the friction cone for the frictionless case. By specializing (2.27) for the case where $\tilde{\mu} = 0$, we obtain that

$$\tilde{n} c_n + \tilde{\nu} c_\nu = 0,\ c_n \geq 0 \Rightarrow c_n = 0,\ c_\nu = 0. \tag{2.28}$$

Using duality in the same way we did to uncover the relationship between MFCQ and (2.25), we can determine that this description is equivalent to the joint constraint matrix $\tilde{\nu}$ having linearly independent columns and

$$\exists v \text{ such that } \tilde{\nu}^T v = 0 \text{ and } \tilde{n}^T v > 0.$$

The latest condition means that the rigid multibody configuration can be disassembled [6]: there exists an external force that breaks all contacts while keeping feasibility of the joint constraints. This condition can be estimated visually for most simple configurations.

The next result makes a connection between the pointed friction cone condition and MFCQ for (2.13) and (2.14).

Lemma 2.5. *If the friction cone of the current configuration is pointed, then the quadratic program (2.13) satisfies MFCQ.*

Proof Assume that, at some point v, (2.13) does not satisfy MFCQ. Then, from (2.25) it follows that there exist the multipliers $\eta_k^{(j)} \geq 0$, $j \in \mathcal{A}$, $k = 1, 2, \ldots, m_C^{(j)}$ and $c_\nu^{(i)}$, $i = 1, 2, \ldots, p$, not all 0, such that

$$0 = \sum_{j \in \mathcal{A}} \sum_{k=1}^{m_C^{(j)}} \eta_k^{(j)} \left(n^{(j)} + \mu^{(j)} d_k^{(j)} \right) + \sum_{i=1,2,\ldots,p} c_\nu^{(i)} \nu^{(i)}. \tag{2.29}$$

Define now, for $j \in \mathcal{A}$,

$$\begin{aligned} c_n^{(j)} &= \sum_{k=1}^{m_C^{(j)}} \eta_k^{(j)} \geq 0 \\ \beta_k^{(j)} &= \mu^{(j)} \eta_k^{(j)} \geq 0 \\ \beta^{(j)} &= \left[\beta_1^{(j)}, \beta_2^{(j)}, \ldots, \beta_{m_C^{(j)}}^{(j)} \right]^T . \end{aligned}$$

One can immediately see with this definition that, for any $j \in \mathcal{A}$,

$$\mu^{(j)} c_n^{(j)} = \sum_{k=1}^{m_C^{(j)}} \beta_k^{(j)} = \mu^{(j)} {e^{(j)}}^T \beta^{(j)} = \mu^{(j)} \left| \left| \beta^{(j)} \right| \right|_1 .$$

Therefore, $c_n = \left\{ c_n^{(j)} \right\}_{j \in \mathcal{A}}$, $\tilde{\beta} = \left\{ \beta^{(j)} \right\}_{j \in \mathcal{A}}$ and $c_\nu = \left\{ c_\nu^{(i)} \right\}_{i=1,2,\ldots,p}$ satisfy the inequalities defining the friction cone $FC(q)$ (2.26), are not all 0 (from our choice of η and c_ν), and from (2.29) satisfy

$$0 = \tilde{n} c_n + \tilde{D} \tilde{\beta} + \tilde{\nu} c_\nu .$$

This contradicts the assumption that the cone is pointed and hence proves the claim. $\diamond$

3. Stability Results for Quadratic Programming

In this section, we bound the size of the solution of the quadratic program (2.24) and, subsequently, (2.13) with respect to the size of the free term of their constraints. Using Theorem 2.1 this will allow us to bound the size of the velocity solution of (2.10)–(2.11) as a function of the terms that depend on the geometrical constraints: Δ and Υ.

Lemma 3.1. *Let $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{n \times p}$ of full column rank, $f \in \mathbb{R}^m$, and $d \in \mathbb{R}^n$, such that $||d|| = 1$, $f_i < 0$ for $i = 1, 2, \ldots, m$, and*

$$A^T d = f < 0, \; B^T d = 0. \tag{3.30}$$

We define

$$\hat{f} = \min_{i=1,m} \{-f_i\} > 0.$$

We denote by $\sigma_0(B)$ the smallest singular value of B, which must be positive from the full rank assumption on B. Then there exists a parameter $c(A, B) > 0$ that depends only on $\hat{f}$, $\sigma_0(B)$, and $||A||$ such that

$$\left| \left| (a^T, b^T) \right| \right| \leq c(A, B) \left| \left| Aa + Bb \right| \right| , \quad \forall a \in \mathbb{R}^m \geq 0, \; \forall b \in \mathbb{R}^p .$$

Proof Let $a \in \mathbb{R}^m \geq 0$, $b \in \mathbb{R}^p$ and define

$$g = Aa + Bb,$$

and multiply the equality by d^T to get

$$d^T g = d^T A a + d^T B b.$$

From the definition of d, the above equality becomes

$$d^T g = f^T a.$$

Since $f < 0$ and $a \geq 0$, we have that $0 \geq -\hat{f}\, ||a||_1 \geq f^T a$. Using the Cauchy-Schwarz inequality in the last equality, together with the fact that $||d|| = 1$, we obtain that

$$||g|| \geq \|d^T g\| \geq \hat{f}\, ||a||_1\, ,$$

that is,

$$||a||_1 \leq \frac{||g||}{\hat{f}}. \tag{3.31}$$

From the definition of g, we have that $Bb = g - Aa$. Taking the 2 norms, and using the triangle inequality, we obtain, after applying the inequality $||a|| \leq ||a||_1$ and (3.31), that

$$||Bb|| \leq ||g|| + ||A||\, ||a|| \leq ||g|| + ||A||\, ||a||_1 \leq ||g|| + ||A||\, \frac{||g||}{\hat{f}}. \tag{3.32}$$

On the other hand, from the definition of the smallest singular value, and since B is full column rank, we have that $||Bb|| \geq \sigma_0(B)\, ||b||$. Using this inequality in (3.32), we obtain that

$$||b|| \leq \frac{1}{\sigma_0(B)}\, ||g|| + \frac{||g||}{\sigma_0(B)\hat{f}}\, ||A||\, .$$

Using the last inequality, (3.31), the inequality $||a|| \leq ||a||_1$ and Minkowski's inequality, we obtain that

$$\left|\left|\left(a^T, b^T\right)\right|\right| \leq ||a|| + ||b|| \leq ||g|| \left(\frac{1}{\hat{f}} + \frac{1}{\sigma_0(B)} + \frac{||A||}{\sigma_0(B)\hat{f}} \right).$$

The conclusion follows after taking

$$c(A,B) = \frac{1}{\hat{f}} + \frac{1}{\sigma_0(B)} + \frac{||A||}{\sigma_0(B)\hat{f}}.$$

$\diamond$

Consider now the quadratic program

$$\begin{array}{rl} \min_v & \frac{1}{2} v^T Q v + k^T v \\ \text{subject to} & A^T v + \alpha \geq 0, \\ & B^T v + \beta = 0, \end{array} \tag{3.33}$$

where $Q \in I\!R^{n \times n}$ is a symmetric positive definite matrix, $A \in I\!R^{n \times m}$, $B \in I\!R^{n \times p}$, $\alpha \in I\!R^{m}$, and $\beta \in I\!R^{p}$. Note that the quadratic program (3.33) satisfies MFCQ if and only if it satisfies (3.30). Therefore, if the quadratic program (3.33) satisfies MFCQ, then there exists $c(A,B) > 0$ with the properties specified in Lemma 3.1.

We perform the change of variable $\overline{v} = Q^{\frac{1}{2}} v$ and transform the quadratic program (3.33) into

$$\begin{array}{rl} \min_{\overline{v}} & \frac{1}{2} \overline{v}^T \overline{v} + \overline{k}^T \overline{v} \\ \text{subject to} & \overline{A}^T \overline{v} + \alpha \geq 0, \\ & \overline{B}^T \overline{v} + \beta = 0, \end{array} \tag{3.34}$$

where

$$\overline{k} = Q^{-\frac{1}{2}} k,\ \overline{A} = Q^{-\frac{1}{2}} A,\ \overline{B} = Q^{-\frac{1}{2}} B.$$

It is immediate that (3.33) satisfies MFCQ if and only if (3.34) satisfies MFCQ.

Theorem 3.2. *Assume that the quadratic programs (3.33) and (3.34) satisfy MFCQ and thus (3.30). Then the quadratic program (3.33) is feasible, and its solution v^* satisfies*

$$\left|\left|Q^{\frac{1}{2}}v^*\right|\right|^2 \leq \left|\left|Q^{-\frac{1}{2}}k\right|\right|^2 + \widehat{c}(A,B,Q)^2\left|\left|\left(\alpha_-^T,\beta^T\right)\right|\right|^2 \leq \left|\left|Q^{-\frac{1}{2}}k\right|\right|^2 + c(A,B,Q)^2\left|\left|\left(\alpha_-^T,\beta^T\right)\right|\right|_\infty^2.$$

Here α_-, with entries $\alpha_{-,i} = -\min\{\alpha_i, 0\}$, $i = 1, 2, \ldots, m$, is the negative part of the vector α, $\widehat{c}(A,B,Q) = c(\overline{A},\overline{B}) = c(Q^{-\frac{1}{2}}A, Q^{-\frac{1}{2}}B)$, where $c(\cdot,\cdot)$ is the parameter obtained in Lemma 3.1 for the quadratic program (3.34), and $c(A,B,Q) = \sqrt{m+n}\,\widehat{c}(A,B,Q)$.

Proof Feasibility of (3.33) and (3.34) when MFCQ holds follows from Lemma 2.3. We write the optimality conditions for (3.34), at the solution $\overline{v}^* = Q^{\frac{1}{2}}v^*$, with Lagrange multipliers $\eta \in I\!R^m$, $\eta \geq 0$, and $\zeta \in I\!R^n$:

$$\begin{array}{rcl} \overline{v}^* + \overline{k} & = & \overline{A}\eta + \overline{B}\zeta, \\ \overline{A}^T\overline{v}^* + \alpha \geq 0 & \perp & \eta \geq 0, \\ \overline{B}^T\overline{v}^* + \beta = 0. & & \end{array} \tag{3.35}$$

Using Lemma 3.1, which applies because, by our assumption, MFCQ and, thus, (3.30), hold for (3.33) and (3.34), we obtain that

$$\left|\left|\left(\eta^T,\zeta^T\right)\right|\right| \leq c(\overline{A},\overline{B})\left|\left|\overline{v}^* + \overline{k}\right|\right|. \tag{3.36}$$

Here $c(\cdot,\cdot)$ is the parameter from Lemma 3.1. Multiplying the first equation of (3.35) by $\overline{v}^*$, and using the last two equations of (3.35), we obtain that

$$\overline{v}^{*T}\left(\overline{v}^* + \overline{k}\right) = \overline{v}^{*T}\overline{A}\eta + \overline{v}^{T*}\overline{B}\zeta = -\eta^T\alpha - \zeta^T\beta.$$

Since $\eta \geq 0$, we have that $\eta^T\alpha \geq -\eta^T\alpha_-$. Using this inequality and the Cauchy-Schwarz inequality in the last equality, we obtain that

$$\overline{v}^{*T}\left(\overline{v}^* + \overline{k}\right) \leq \eta^T\alpha_- - \zeta^T\beta \leq \left|\left|\left(\alpha_-^T,\beta^T\right)\right|\right|\left|\left|(\eta^T,\zeta^T)\right|\right|.$$

Using (3.36), we obtain that

$$\overline{v}^{*T}\left(\overline{v}^* + \overline{k}\right) \leq c(\overline{A},\overline{B})\left|\left|\left(\alpha_-^T,\beta^T\right)\right|\right|\left|\left|\overline{v}^* + \overline{k}\right|\right|. \tag{3.37}$$

Using that $\overline{k} = \overline{v}^* + \overline{k} - \overline{v}^*$ and the inequality (3.37), we obtain that

$$\begin{array}{rcl} \left|\left|\overline{k}\right|\right|^2 & = & \left|\left|\overline{v}^*\right|\right|^2 + \left|\left|\overline{v}^* + \overline{k}\right|\right|^2 - 2\overline{v}^{*T}\left(\overline{v}^* + \overline{k}\right) \geq \\ & & \left|\left|\overline{v}^*\right|\right|^2 + \left|\left|\overline{v}^* + \overline{k}\right|\right|^2 - 2c(\overline{A},\overline{B})\left|\left|\left(\alpha_-^T,\beta^T\right)\right|\right|\left|\left|\overline{v}^* + \overline{k}\right|\right| = \\ & & \left|\left|\overline{v}^*\right|\right|^2 - c(\overline{A},\overline{B})^2\left|\left|\left(\alpha_-^T,\beta^T\right)\right|\right|^2 + \left|\left|\left|\left|\overline{v}^* + \overline{k}\right|\right| - c(\overline{A},\overline{B})\left|\left|\left(\alpha_-^T,\beta^T\right)\right|\right|\right|\right|^2 \geq \\ & & \left|\left|\overline{v}^*\right|\right|^2 - c(\overline{A},\overline{B})^2\left|\left|\left(\alpha_-^T,\beta^T\right)\right|\right|^2. \end{array}$$

Since $\overline{v}^* = Q^{\frac{1}{2}}v$ and $\overline{k} = Q^{-\frac{1}{2}}k$, the first inequality in the conclusion of the theorem follows immediately. The second inequality follows by using the inequality between $||\cdot||_2$ and $||\cdot||_\infty$. $\diamond$

Corollary 3.3. *Assume that the quadratic program (2.13) satisfies MFCQ. Let $A(q^{(l)},\tilde{\mu})$ define the matrix of the inequality constraints and $B(q^{(l)})$ define the matrix of the equality constraints in (2.13). Further, assume that $\Gamma^{(j)} \geq 0$, $j \in \mathcal{A}$. Let $c(A(q^{(l)},\tilde{\mu}), B(q^{(l)}), M^{(l)})$ be the quantity defined in Theorem 3.2. Then the solution $v^{(l+1)}$ of (2.13) satisfies*

$$v^{(l+1)^T}M^{(l)}v^{(l+1)} \leq v^{(l)}M^{(l)}v^{(l)} + h_l^2k^{(l)}M^{(l)-1}k^{(l)} + 2h_lv^{(l)^T}k^{(l)} + c(A(q^{(l)},\tilde{\mu}), B(q^{(l)}), M^{(l)})^2\left|\left|\Delta_-^{(l)},\Upsilon^{(l)}\right|\right|_\infty^2.$$

Note The assumption $\Gamma^{(j)} \geq 0$, $j \in \mathcal{A}$, is automatically satisfied under the conditions of Theorem 2.1.

Proof Since $\Gamma^{(j)} \geq 0$, $j \in \mathcal{A}$, we obtain that $(\Delta^{(j)} + \Gamma^{(j)})_- \leq \Delta_-^{(j)}$, for $j \in \mathcal{A}$.

Since MFCQ holds for (2.13), we can apply Theorem 3.2 to the quadratic program (2.13), with $v^* = v^{(l+1)}$, $Q = M^{(l)}$, $k = \widehat{k} = -Mv^{(l)} - h_l k^{(l)}$, $\beta = \Upsilon$, $\alpha = \Gamma + \Delta$. Using the inequality in the preceding paragraph, we obtain that

$$\left|\left|M^{(l)\frac{1}{2}} v^{(l+1)}\right|\right|^2 \leq \left|\left|\; M^{(l)-\frac{1}{2}}\left(M^{(l)}v^{(l)} + h_l k^{(l)}\right)\;\right|\right|^2 + c\left(A(q^{(l)}, \tilde{\mu}), B(q^{(l)}), M^{(l)}\right)^2 \left|\left|(\Delta_-^T, \Upsilon^T)\right|\right|_\infty^2 .$$

Working on the term following the inequality sign, we obtain that

$$\left|\left|\; M^{(l)-\frac{1}{2}}\left(M^{(l)}v^{(l)} + h_l k^{(l)}\right)\;\right|\right|^2 \quad = \quad {v^{(l)}}^T M^{(l)} v^{(l)} + h_l^2 {k^{(l)}}^T M^{(l)-1} k^{(l)} + 2h_l {v^{(l)}}^T k^{(l)}.$$

This proves the claim since we also have that

$$\left|\left|M^{(l)\frac{1}{2}} v^{(l+1)}\right|\right|^2 = {v^{(l+1)}}^T M^{(l)} v^{(l+1)}.$$

$\diamond$

4. Constraint Stability Results

Before we can state our results, we need to ensure certain regularity properties of the mappings that are used to define the noninterpenetration constraints (2.3). We now describe in some detail how the mappings $\Phi^{(j)}$ are defined and some of their properties.

We denote by n_{bod} the number of rigid bodies in the system. Let $Q_1, Q_2, \ldots, Q_{n_{bod}}$ be the spaces that contain the generalized coordinates of the bodies $B_1, B_2, \ldots B_{n_{bod}}$, whose generalized coordinates we denote by $b_1, b_2, \ldots, b_{n_{bod}}$. These spaces are locally homeomorphic with some bounded open set of R^s [21].

The aggregate generalized position (from here on, the generalized position) becomes $q = (b_1^T, b_2^T, \ldots, b_{n_{bod}}^T)^T$. We denote $Q = Q_1 \times Q_2 \times \ldots Q_{n_{bod}}$. At a generalized position vector q, we denote by $\delta_{i_1 i_2}(q)$ the signed distance between the bodies B_{i_1} and B_{i_2} [18]:

$$\delta_{i_1 i_2}(q) = \begin{cases} \text{Euclidean distance between } B_{i_1} \text{ and } B_{i_2} \text{ if they do not interpenetrate.} \\ \text{Minus the length of smallest relative translation vector that separates } B_{i_1} \text{ and } B_{i_2} \\ \text{if they are in an interpenetrating configuration.} \end{cases} \tag{4.38}$$

Since in the following we work only with the signed distance function, we refer to $\delta_{i_1 i_2}(q)$ simply as the *distance function*. This distance is a mapping that depends continuously on q and on the shape of the bodies, but we consider the latter dependency only implicitly.

The feasible set of the noninterpenetration constraint B_{i_1} *cannot interpenetrate* B_{i_2} is $\delta_{i_1 i_2}(q) \geq 0$. The feasible set of all noninterpenetration and joint constraints is then defined by

$$\delta_{i_1 i_2}(q) \geq 0,\ 1 \leq i_1 < i_2 \leq n_{bod}, \quad \Theta^{(i)}(q) = 0,\ i = 1, 2, \ldots, m. \tag{4.39}$$

To rewrite the noninterpenetration constraints in the framework of Section 2 and of [1, 2, 3, 29], we associate a pair (i_1, i_2) with an index $j \in \{1, 2, \ldots, p\}$ and define $\Phi^{(j)}(q) = \delta_{i_1 i_2}(q)$. Here

$p = \frac{n(n+1)}{2}$. In the following analysis we use (2.3) and (2.1) interchangeably with (4.39) to describe the geometrical constraints.

In proving that our method provides constraint stabilization, we need differentiability of $\delta_{i_1 i_2}(q)$, $1 \le i_1 < i_2 \le n_{bod}$, over a sufficiently large subset of Q. Unfortunately, differentiability of $\delta_{i_1 i_2}(q)$, $1 \le i_1 < i_2 \le n_{bod}$, over all of Q cannot be assumed even for simple shapes, as can be seen in Figure 1. In this figure we have an immovable infinite beam of constant height H (body 1) and a disk of radius R (body 2).

Since the beam is immovable, the generalized coordinates defining the position of the disk are $q = (x, y, \phi)$, where ϕ is the angle that defines the rotation of the disk with respect to the world. One can immediately see that the distance between the disk and the beam is a function of only y and is equal to

$$\delta_{12}(q) = |y| - R - \frac{H}{2} \tag{4.40}$$

and is not a differentiable function of y. It is therefore necessary to create a stabilization framework that works with weaker differentiability assumptions.

Since we are interested in achieving feasibility as the time step goes to 0 and keeping infeasibility under control for finite time step, it will be sufficient for $\delta_{i_1 i_2}(q)$ to be differentiable only for small values of the interpenetration for fixed values of i_1 and i_2.

To formally accommodate this requirement, we define, for some $\epsilon > 0$, the set

$$\Omega_\epsilon = \left\{ q \in Q | \delta_{i_1 i_2}(q) \ge -\epsilon, i_1, i_2 = 1, 2, \ldots, n_{bod}, \; \left| \Theta^{(i)}(q) \right| \le \epsilon, \; i = 1, 2, \ldots, m \right\}.$$

We make the following assumption about the kinematic description of the noninterpenetration constraints.

(A1) There exists $\epsilon_0 > 0$ such that, $\forall 0 \le \epsilon \le \epsilon_0$, we have that $\delta_{i_1 i_2}(q)$ for $1 \le i_1, i_2 \le n_{bod}$ and $\Theta^{(i)}(q)$ for $i = 1, 2, \ldots, m$ are twice continuously differentiable and their first and second derivatives are uniformly bounded over Ω_ϵ by C_1^d and C_2^d, respectively. For the case where there are no joint constraints, this assumption holds if the bodies are strictly convex and smooth, for some $\epsilon_0 > 0$ [5]. It also holds if one body is an infinite flat wall and the others are strictly convex. An analysis of the limits of applicability of the differentiability assumptions can be found in [5].
Since any two-dimensional or three-dimensional body can be approximated by a union of strictly convex bodies with smooth boundary, Assumption (A1) holds, in principle, when replacing bodies of any shape by such approximations while adding the corresponding mappings to the list of those defining (4.39). For efficiency purposes, however, it may be useful to work with bodies whose shape is not smooth but only piecewise smooth, such as box-shaped bodies. We defer the inclusion of bodies with piecewise smooth shape in the analysis to future research.

In addition to this assumption, and to define our approach completely, we need to specify the mechanism by which we choose the active set $\mathcal{A}$. Our definition of the active set depends on a parameter $\hat{\epsilon}$. We define

$$\mathcal{A}(q) = \left\{ (i_1, i_2) | \Phi^{(j)}(q) \le \hat{\epsilon}, \; 1 \le j \le p \right\}. \tag{4.41}$$

We now make the following assumptions concerning the dynamics.

(D1) The mass matrix $M^{(l)} = M(q^{(l)})$ is constant. We denote the constant mass matrix by M. This situation can be achieved by using the Newton-Euler formulation in body coordinates in three dimensions [25]. In two dimensions, the same situation is achieved by using the world coordinates [21].

(D2) Denote by $A(q, \tilde{\mu})$ the matrix defining the inequality constraints and by $B(q)$ the matrix defining the equality constrains of (2.13). Clearly the dimension of $A(q, \tilde{\mu})$ depends on the size of the active set and of the size of the approximation to the friction cone. *We assume* that

$$c(A(q, \tilde{\mu}), B(q), M) \leq c_0, \ \forall \epsilon \in [0, \epsilon_0], \ \forall q \in \Omega_\epsilon(q),$$

where $c(\cdot, \cdot, \cdot)$ is the parameter defined in Theorem 3.2.
Based on Theorem 3.2, the existence of $c(\cdot, \cdot, \cdot)$ is guaranteed by (2.13) satisfying MFCQ. In turn, from Lemma 2.5, (2.13) satisfies MFCQ if the pointed friction cone assumption is satisfied at the current configuration (position $q^{(l)}$ and active friction coefficients $\tilde{\mu}$). Therefore Assumption (D2) is implied by an uniformly pointed friction cone assumption. The uniformly pointed friction cone assumption has been used in the past and is an essential condition in proving convergence of a related time-stepping scheme as $h \to 0$ [30]. Based on the disassemblability interpretation following Theorem 2.4, one can show that, if all the bodies are strictly convex and smooth, and no joints are involved, then this assumption holds for ϵ_0 and $\tilde{\mu}$ sufficiently small.
We also have a quantitative description of conditions that imply Assumption (D2). From Lemma 3.1 and Theorem 3.2 Assumption (D2) holds if $m, n, ||A(q, \tilde{\mu})||$ are uniformly upper bounded and if $\hat{f}$, which exists from Lemma 2.5 when the friction cone is pointed, and $\sigma_0(B(q))$ are uniformly lower bounded over Ω_ϵ.

(D3) The external force is continuous and increases at most linearly with the position and the velocity and is uniformly bounded in time. Hence,

$$k(t, v, q) = k_0(t, v, q) + f_c(v, q) + k_1(v) + k_2(q), \tag{4.42}$$

and that there exists $c_K \geq 0$ such that

$$||k_0(t, v, q)|| \leq c_K, \ ||k_1(v)|| \leq c_K \, ||v|| \, , \ ||k_2(q)|| \leq c_K \, ||q|| \, . \tag{4.43}$$

Here $f_c(v, q)$ is the Coriolis force, which satisfies the following important property [2]:

$$v^T f_c(v, q) = 0, \ \forall v, q. \tag{4.44}$$

Elastic forces are contained in k_2, whereas damping forces are contained in k_1.

Our main result concerns the behavior of the infeasibility of the noninterpenetration and the joint constraint. We define the measure of constraint infeasibility:

$$I(q) = \max_{1 \leq j \leq p, 1 \leq i \leq m} \left\{ \Phi_-^{(j)}(q), |\Theta^{(i)}(q)| \right\}. \tag{4.45}$$

We also define a measure of infeasibility that is attached to a choice of the active set:

$$I^{\mathcal{A}}(q) = \max_{j \in \mathcal{A}, 1 \leq i \leq m} \left\{ \Phi_-^{(j)}(q), |\Theta^{(i)}(q)| \right\}. \tag{4.46}$$

Since, by the definition (4.41), the active set contains all noninterpenetration constraints that are infeasible at the current point q, we must have that

$$I(q) = I^{\mathcal{A}(q)}(q).$$

In general, however, we will have, for different q_1 and q_2, that

$$I(q_1) \neq I^{\mathcal{A}(q_2)}(q_1).$$

Looking at the definition of Δ and Υ after (2.10)–(2.11), we get that

$$I^{\mathcal{A}}(q^{(l)}) = h_l \left|\left|\Delta_-^{(l)^T}, \Upsilon^{(l)}\right|\right|. \tag{4.47}$$

A connection between Ω_ϵ for ϵ_0 and $I(q)$ is also that

$$q \in \Omega_\epsilon \Leftrightarrow I(q) \leq \epsilon.$$

We will prove our result for a more general form of the algorithm than the one provided by (2.10)–(2.11), to also incorporate time-stepping methods that are based on suitable relaxations of the potentially nonconvex subproblem (2.10)–(2.11) [1]:

- Given $q^{(l)}, v^{(l)}, h_l$, determine $v^{(l+1)}$, which is a solution of (2.13), where $\mathcal{A}$ is defined as in (4.41) and where all the relevant data, with the exception of $\Gamma^{(j)}, j \in \mathcal{A}$, are defined as in the setup of (2.10)–(2.11). *Assume* that the solution $v^{(l+1)}$ and Γ have the property that $\Gamma \geq 0$ and

$$\forall j \in \mathcal{A}, \exists i, 1 \leq i \leq m_C^{(j)} \text{ such that } -\mu^{(j)} d_i^{(j)^T} v^{(l+1)} \geq \Gamma^{(j)}. \tag{4.48}$$

  This assumption is satisfied in at least two cases:

  1. When $v^{(l+1)}$ is found by solving (2.10)–(2.11) and $\Gamma = \tilde{\mu}\lambda^*$, where λ^* is the quantity from Lemma 2.2. This case follows from Theorem 2.1 and Lemma 2.2. When the friction cone is pointed, as we will assume by invoking Assumption (D2), the existence of velocity $v^{(l+1)}$ guaranteed by Theorem 2.4.
  2. When (2.13) is solved directly after choosing $\Gamma = 0$. This case corresponds to the convex relaxation algorithm described in [1]. When the friction cone is pointed, as we will assume by invoking Assumption (D2), the existence of a solution of (2.13) is guaranteed by using, successively, the Lemmas 2.5 and 2.3.

- Compute $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$, take $l = l + 1$, and restart.

Theorem 4.1. *Consider the time-stepping algorithm defined above with the choice of active set defined by (4.41). The algorithm is applied over a finite time interval* $[0, T]$*, and the time steps* $0 < h_l$ *satisfy*

$$\sum_{i=0}^{N-1} h_l = T \quad \textit{and} \quad \frac{h_{l-1}}{h_l} \leq c_h, \; l = 1, 2, \ldots, N-1.$$

In addition, it is assumed that the system satisfies the assumptions (A1) and (D1)–(D3) and that the system is initially feasible, that is, $I(q^{(0)}) = 0$.

Then, there exist $H > 0$*,* $V > 0$*, and* $C_c > 0$ *such that, whenever, in addition to the requirements above, we have that* $h_l < H$, $\forall\, l$, $0 \leq l \leq N-1$, *we will also have that*

1. $\left|\left|v^{(l)}\right|\right| \leq V$, $\forall 1 \leq l \leq N$ *and*
2. $I(q^{(l)}) \leq C_c \left|\left|v^{(l)}\right|\right|^2 h_{l-1}^2$, $\forall 1 \leq l \leq N$.

Note The proof of the Theorem is considerably lengthened by the fact that we have to ensure that $q^{(l)} \in \Omega_\epsilon$, for $l = 1, 2, \ldots, N$ and for an appropriately chosen value of ϵ, in order to be able to use Assumption (A1) and, subsequently, to apply Taylor's Theorem. As we argued before defining Assumption (A1), it would be unrealistic to assume the everywhere differentiability of the mapping defining the noninterpenetration constraint (2.3), which would substantially simplify the proof.

Proof We will prove this theorem by showing that all conditions in Theorem II.2 are met. To that end, we make the following identification:

$$z_l = \left|\left|M^{\frac{1}{2}} v^{(l)}\right|\right|, \quad w_l = \left|\left|q^{(l)}\right|\right|, \quad \theta^{(l)} = I(q^{(l)}). \tag{4.49}$$

We use the assumption (D1) that the mass matrix M is constant.

Assume now that for some $l \in \{0, 1, \ldots, N\}$ we have that $q^{(l)} \in \Omega_{\frac{\epsilon_0}{2}}$, that is,

$$\theta_l = I(q^{(l)}) \leq \frac{\epsilon_0}{2}. \tag{4.50}$$

To obtain the inequalities from the statement of Theorem II.2, we apply Corollary 3.3, which applies because of our assumption (D2) and (4.50). Before applying it, we want to obtain an upper bound on the term $v^{(l)^T} k^{(l)}$, based on the assumption (D3). Using the identification (4.49) and the property of the Coriolis force (4.44), we obtain that

$$\begin{aligned} v^{(l)^T} k^{(l)} &= v^{(l)^T} \left(f_c(v^{(l)}, q^{(l)}) + k_0(v^{(l)}, q^{(l)}, t^{(l)}) + k_1(v^{(l)}) + k_2(q^{(l)}) \right) \\ &= v^{(l)^T} \left(k_0(v^{(l)}, q^{(l)}, t^{(l)}) + k_1(v^{(l)}) + k_2(q^{(l)}) \right) \\ &\leq c_K \left|\left|M^{-1}\right|\right|^{\frac{1}{2}} z_l + c_K \left|\left|M^{-1}\right|\right| z_l^2 + c_K \left|\left|M^{-\frac{1}{2}}\right|\right| z_l w_l. \end{aligned} \tag{4.51}$$

We denote

$$\psi_1(z, w) = \max_{t \leq T,\ ||v|| \leq z \left|\left|M^{-\frac{1}{2}}\right|\right|, ||q|| \leq w} k(q, v, t)^T M^{-1} k(q, v, t), \tag{4.52}$$

which is a continuous function, following our assumption (D3). Using now the conclusion of the Corollary 3.3 and the parameter c_0 defined in the assumption (D2), together with the identification (4.49), and the bounds (4.51) and (4.52), we obtain that

$$z_{l+1}^2 \leq z_l^2 + h_l 2 \left(c_K \left|\left|M^{-1}\right|\right|^{\frac{1}{2}} z_l + c_K \left|\left|M^{-1}\right|\right| z_l^2 + c_K \left|\left|M^{-\frac{1}{2}}\right|\right| z_l w_l \right) + h_l^2 \psi_1(z, w) + c_0 \frac{||\theta_l||^2}{h_l^2}, \tag{4.53}$$

where the substitution of θ_l in the last term is possible from the relations (4.46) and (4.47) and since $I(q) = I^{\mathcal{A}}(q)(q)$.

For the positions $q^{(l)}$ we have, from the definition of the time-stepping scheme, that $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$, which, using (4.49), leads to

$$w_{l+1} \leq w_l + h_l \left|\left|M^{-\frac{1}{2}}\right|\right| z_l. \tag{4.54}$$

We have thus obtained that

$$\boxed{(4.50) \Rightarrow (4.53) \text{ and } (4.54).}$$

We now show that if (4.50) holds, then

$$C_1^d h_l \left|\left|v^{(l+1)}\right|\right| \leq \frac{1}{2}\min\{\hat{\epsilon}, \frac{\epsilon_0}{2}\} \Rightarrow I(q^{(l)} + \tau v^{(l+1)}) \leq \epsilon_0,\ \forall \tau \in [0, h_l]. \tag{4.55}$$

Indeed, assume that (4.55) does not hold. Define

$$t^* = \min\left\{t | I(q^{(l)} + \tau v^{(l+1)}) \leq \epsilon_0,\ \forall 0 \leq \tau \leq t\right\}.$$

From (4.50) we have that $t^* > 0$, and, since (4.55) does not hold, we must have that $t^* < h_l$. From the definition of (4.45) we have that there exists $1 \leq i_1, i_2 \leq n_{bod}$, or $1 \leq i \leq m$, such that either $\delta_{i_1,i_2}(q^{(l)} + t^* v^{(l+1)}) = -\epsilon_0$ or $|\Theta^{(i)}(q^{(l)} + t^* v^{(l+1)})| = \epsilon_0$. We assume the former, the latter case following much the same way. From (4.50) we must have that $\delta_{i_1 i_2}(q^{(l)}) \geq -\frac{\epsilon_0}{2}$. Since, from the definition of t^*, we have that $I(q^{(l)} + \tau v^{(l+1)}) \leq \epsilon_0$, $\forall 0 \leq \tau \leq t^*$, we can apply Assumption (A1) and Taylor's theorem to obtain that

$$\frac{\epsilon_0}{2} \leq \delta_{i_1 i_2}(q^{(l)}) - \delta_{i_1 i_2}(q^{(l)} + t^* v^{(l+1)}) \leq C_1^d h_l \left|\left|v^{(l+1)}\right|\right| \leq \frac{\epsilon_0}{4},$$

which is a contradiction. This shows that (4.55) must hold.

We also show that if (4.50) holds, then

$$C_1^d h_l \left|\left|v^{(l+1)}\right|\right| \leq \frac{1}{2}\min\{\hat{\epsilon}, \frac{\epsilon_0}{2}\} \Rightarrow \delta_{i_1,i_2}(q^{(l+1)}) \geq \frac{\hat{\epsilon}}{2} > 0,\ \ \forall (i_1, i_2) = j \notin \mathcal{A}(q^{(l)}). \tag{4.56}$$

Indeed, if $(i_1, i_2) \notin \mathcal{A}(q^{(l)})$, then, following the definition of the active set (4.41), we have that $\delta_{i_1,i_2}(q^{(l)}) \geq \hat{\epsilon}$. Using now (4.55), Assumption (A1), and Taylor's theorem, we obtain that

$$\delta_{i_1,i_2}(q^{(l+1)}) \geq \delta_{i_1,i_2}(q^{(l)}) - C_1^d h_l \left|\left|v^{(l+1)}\right|\right| \geq \hat{\epsilon} - \frac{\hat{\epsilon}}{2} > 0,$$

which proves (4.56).

Using the identification (4.49), we obtain that $\left|\left|v^{(l+1)}\right|\right| \leq \left|\left|M^{-\frac{1}{2}}\right|\right| z_{l+1}$ and thus that

$$\begin{aligned} C_1^d h_l \left|\left|M^{-\frac{1}{2}}\right|\right| z_{l+1} &\leq \frac{1}{2}\min\{\hat{\epsilon}, \frac{\epsilon_0}{2}\} \\ \Longrightarrow C_1^d h_l \left|\left|v^{(l+1)}\right|\right| &\leq \frac{1}{2}\min\{\hat{\epsilon}, \frac{\epsilon_0}{2}\}. \end{aligned} \tag{4.57}$$

Using (4.56) and (4.55), we now obtain that

$$\text{(4.50) and (4.57)} \Longrightarrow \begin{cases} I(q^{(l)} + \tau v^{(l+1)}) \leq \epsilon_0,\ \forall \tau \in [0, h_l] \\ \delta_{i_1 i_2}(q^{(l+1)}) \geq \frac{\hat{\epsilon}}{2} > 0,\ \ \forall (i_1, i_2) = j \notin \mathcal{A}(q^{(l)}). \end{cases} \tag{4.58}$$

Finally we show that if (4.50) and (4.57) hold, then the following relation holds:

$$\theta_{l+1} = I(q^{(l+1)}) \leq \frac{1}{2} C_2^d h_l^2 \left|\left|M^{-1}\right|\right| z_{l+1}^2. \tag{4.59}$$

To prove this statement, we have three cases to consider, following the definition of (4.45):

1. $I(q^{(l+1)}) = -\delta_{(i_1,i_2)}(q^{(l+1)}) = -\Phi^{(j)}(q^{(l+1)})$, where $j = (i_1, i_2) \in \mathcal{A}(q^{(l)})$. By the definition of the algorithm, and since $v^{(l+1)}$ is a solution of (2.13), we must have that
$$n^{(j)^T} v^{(l+1)} + \mu^{(j)} d_i^{(j)^T} v^{(l+1)} \geq -\left(\Gamma^{(j)} + \Delta^{(j)}\right),\ j \in \mathcal{A},\ i \in m_C^{(j)}.$$
Using the assumption (4.48), we can assume that for any $j \in \mathcal{A}$ there is an $i \in \{1, 2, \ldots, m_C^{(j)}\}$ such that $-\mu^{(j)} d_i^{(j)^T} v^{(l+1)} \geq \Gamma^{(j)}$. For that particular i, we obtained from the displayed inequality that
$$n^{(j)^T} v^{(l+1)} \geq -\Delta^{(j)} = -\frac{\Phi^{(j)}(q^{(l)})}{h_l}.$$
Using the last inequality, (2.4), and Taylor's theorem, which applies as a result of Assumption (A1) and (4.58) and since $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$, we have that
$$\Phi^{(j)}(q^{(l+1)}) \geq \Phi^{(j)}(q^{(l)}) + h_l \nabla_q \Phi^{(j)^T} v^{(l+1)} - \frac{1}{2} C_2^d h_l^2 \left|\left|v^{(l+1)}\right|\right|^2 \geq -\frac{1}{2} C_2^d h_l^2 \left|\left|v^{(l+1)}\right|\right|^2,$$
which, from (4.49), implies (4.59). Here we used the fact that $n^{(j)} = \nabla_q \Phi^{(j)}(q^{(l)})$.
2. $I(q^{(l+1)}) = -\delta_{i_1 i_2}(q^{(l+1)}) = -\Phi^{(j)}(q^{(l+1)}) > 0$, where $j = (i_1, i_2) \notin \mathcal{A}(q^{(l)})$. From (4.58) we have that $\Phi^{(j)}(q^{(l+1)}) \geq \frac{\hat{\epsilon}}{2}$, so this case cannot occur.
3. $I(q^{(l+1)}) = |\Theta^{(i)}(q^{(l+1)})|$, where $1 \leq i \leq m$. By Theorem 2.1, and since $v^{(l+1)}$ is a solution of (2.13), we must have that
$$\nu^{(i)^T} v = -\Upsilon^{(i)} = -\frac{\Theta^{(i)}(q^{(l)})}{h_l}.$$
Using the last equality, (2.2), and Taylor's theorem, which applies as a result of Assumption (A1) and (4.58) and since $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$, we have that
$$|\Theta^{(i)}(q^{(l+1)})| \leq |\Theta^{(i)}(q^{(l)}) + h_l \nabla_q \Theta^{(i)^T} v^{(l+1)}| + \frac{1}{2} C_2^d h_l^2 \left|\left|v^{(l+1)}\right|\right|^2 \leq \frac{1}{2} C_2^d h_l^2 \left|\left|v^{(l+1)}\right|\right|^2$$
which, from (4.49), implies (4.59). Here we used the fact that $\nu^{(i)} = \nabla_q \Theta^{(i)}(q^{(l)})$.

We therefore have that
$$\boxed{(4.50) \text{ and } (4.57) \Rightarrow (4.59).}$$

We use the boxed implications and the identification defined in (4.49) for $l = n$, and associate the equations as follows: (4.50) $\leftrightarrow$ (II.63), (4.53) $\leftrightarrow$ (II.64), (4.54) $\leftrightarrow$ (II.65), (4.57) $\leftrightarrow$ (II.66) and (4.59) $\leftrightarrow$ (II.67). After an appropriate choice of the parameters c_i, $i = 1, 2, \ldots, 5$, we can now apply Theorem II.2.

From Theorem II.2 we obtain that there exist an $H > 0$ and $Z > 0$ such that, whenever we have that $h_l \leq H, \forall\, 0 \leq l \leq N$, we obtain that $||z_l|| \leq Z, \forall\, 0 \leq l \leq N$. Defining $V = \left|\left|M^{-\frac{1}{2}}\right|\right| Z$ and using (4.49) we get that $||v_l|| \leq V$, $\forall 0 \leq l \leq N$, which proves Part 1 of the claim. From the second part of Theorem II.2 we obtain that (4.50) and (4.57), and, therefore, (4.59), hold for $l = 1, 2, \ldots, N$. The proof of Part 2 follows after using that $z_{l+1} \leq \left|\left|M^{\frac{1}{2}}\right|\right| \left|\left|v^{(l+1)}\right|\right|$, that follows from (4.49) and from choosing $C_c = \frac{1}{2} C_2^d \left|\left|M^{-1}\right|\right| ||M||$ in (4.59). $\diamond$

Theorem 4.1 has the following important consequences for sufficiently small time steps:

1. Our approach achieves constraint stabilization, which is quantified by the conclusion of Part 2 of the Theorem. Constraint stabilization does not follow from $I(q^{(l)}) \to 0$ as $h^{(l)} \to 0$, which also occurs for unstabilized, convergent schemes. It follows from the fact that the infeasibility is upper bounded by a local measure of the size of the solution. In particular, $v^{(l+1)} = 0 \Rightarrow I(q^{(l+1)}) = 0$, which does not occur for unstabilized schemes.
2. The velocity remains bounded for a finite simulation time interval and for a fairly general form of the external force (4.42).
3. We can use a constant time step and keep the infeasibility under control *while solving only one linear complementarity problem per step.* Therefore the amount of computation is, in some sense, predictable, which is very useful for real-time computation.

5. Applying the Scheme to DAE

Important insight into the behavior of the scheme we propose here can be obtained by applying the scheme to a problem that has only joint constraints. We consider here only the case where the time step is constant, that is, $h_l = h$, $\forall l$.

We compare the method from this work to the similar unstabilized method. A unifying framework is

$$\begin{array}{rcl} Mv^{(l+1)} - \sum_{i=1}^{m} c_\nu^{(i)} \nu^{(i)} & = & Mv^{(l)} + hk^{(l)} \\ {\nu^{(i)}}^T v^{(l+1)} & = & \Upsilon^{(i)},\ i = 1, 2, \ldots, m \\ q^{(l+1)} & = & q^{(l)} + hv^{(l+1)}. \end{array} \tag{5.60}$$

Here the choice $\Upsilon^{(i)} = -\frac{\Theta^{(i)}(q^{(l)})}{h}$, $i = 1, 2, \ldots, m$, corresponds to our method, whereas $\Upsilon^{(i)} = 0$, $i = 1, 2, \ldots, m$ corresponds to the unstabilized method. Using Theorem 4.1, we obtain that $\Upsilon \to 0$ as $h^{(l)} \to 0$. Hence, our method is convergent to the solution of the corresponding DAE.

We compare the two methods for a pendulum with gravity example, and we plot the infeasibility for both methods in Figure 2. The methods were applied with constant time step $h = 0.1$ and run for 1,000 steps, corresponding to the final time $T = 100s$. We see that the velocity drifts off for the unstabilized method, whereas it remains bounded and, in fact, decreases for our method.

Of course, superior methods exist for DAE, which in particular use projection or some other type of constraint stabilization. Nevertheless, our method has the following characteristics:

1. When applied to a differential algebraic equation it solves only one linear system per time step.
2. No additional parameters are needed in order to stabilize the constraints such as the ones described in [10] and [8].

Therefore, to have a fair comparison, we would need to compare our method to a similar method, but none seems to exist in the literature. The postprocessing method [7, 9] solves one additional linear system per step. However, the method can be modified in such a fashion that it does only one overall factorization per step, and it thus needs only one additional backward/forward substitution per step compared to our method. Nevertheless, in the case where the linear system is solved by an iterative technique, the difference may prove significant.

When applied to DAEs, our method can be interpreted as the limit case of another approach. We approximate the effect of the equality constraints by using an elastic force that pushes the configuration toward the feasible manifold. Such an elastic force is generated, for example, by the potential

$$\frac{1}{2}C_k\sum_{i=1}^{m}\left(\Theta^{(i)}(q)\right)^2,$$

where C_k is an appropriately chosen parameter. The force corresponding to this penalty is

$$F_C(q) = -\sum_{i=1}^{m} C_k\Theta^{(i)}(q)\nabla_q\Theta^{(i)}(q),$$

and the problem does not have any additional constraints.

Assume now that we add this force to the given external force $k(t,q,v)$ and we use the linearly implicit Euler method to time step. For that, we would have to compute the Jacobian of $F_C(q)$. To maintain the symmetry of the problem, however, we assume that the external force $k(t,q,v)$ is nonstiff, and we use the following approximated Jacobian for $F_C(q)$:

$$\nabla_q F_C(q) \approx JF_C(q) = -\sum_{i=1}^{m} C_k\nabla_q\Theta(q)^{(i)}\nabla_q\Theta(q)^{(i)^T} = -\sum_{i=1}^{m} C_k\nu^{(i)}(q)\nu^{(i)^T}(q).$$

Here we use the notation $\nu^{(i)} = \nabla_q\Theta(q^{(l)})$ that was introduced at the beginning of this work. The time-stepping scheme becomes

$$Mv^{(l+1)}+h^2\sum_{i=1}^{m} C_k\nu^{(i)}(q^{(l)})\nu^{(i)^T}(q^{(l)})v^{(l+1)} = Mv^{(l)}+hk(t^{(l)},v^{(l)},q^{(l)})-h\sum_{i=1}^{m} C_k\Theta^{(i)}(q^{(l)})\nu^{(i)}(q^{(l)}).$$

The matrix of the linear system is positive definite, which now guarantees the existence of a solution $v^{(l+1)}$ for any $h > 0$. This observation has been used to adapt this method for stiff rigid multibody dynamics with contact and friction [2]. To simplify the notation we now ignore the dependence of all data of the problem on $t^{(l)}$, $v^{(l)}$ and $q^{(l)}$, with the exception of $\Theta^{(i)}$.

In the previous equality, we take the part involving C_k to the left side of the inequality, and we define

$$c_\nu^{(i)} = hC_k\left(\Theta^{(i)} + h\nu^{(i)}v^{(l+1)}\right).$$

Defining $c_\nu^{(i)}$ as a new variable, we obtain the following linear system

$$\begin{array}{rcl} Mv^{(l+1)} - \sum_{i=1}^{m} c_\nu^{(i)}\nu^{(i)} & = & Mv^{(l)} + hk \\ h\nu^{(i)^T}v^{(l+1)} + \Theta^{(i)}(q^{(l)}) - \frac{1}{C_k}c_\nu^{(i)} & = & 0,\ i=1,2,\ldots,m \\ q^{(l+1)} & = & q^{(l)} + hv^{(l+1)}. \end{array} \tag{5.61}$$

Assuming that the solution of this system stays bounded, we obtain, as $C_k \to \infty$, the following system,

$$\begin{array}{rcl} Mv^{(l+1)} - \displaystyle\sum_{i=1}^{m} c_\nu^{(i)}\nu^{(i)} & = & Mv^{(l)} + hk \\ \nu^{(i)^T}v^{(l+1)} + \dfrac{\Theta^{(i)}(q^{(l)})}{h} & = & 0,\ i=1,2,\ldots,m \\ q^{(l+1)} & = & q^{(l)} + hv^{(l+1)}, \end{array} \tag{5.62}$$

which is precisely (5.60). This shows that our method can be seen as the limit case of a penalty-type method that is treated linearly implicitly. This feature explains to some extent why our method can stabilize the constraint behavior and why it does not need to tune any parameter (except, perhaps, the time step) toward that end.

This also shows one possible caveat of the method. If, at some point, one decides that the constraint error is too large, then one may decide to reduce the time step. If aggresive time step reduction is done, the right-hand side of (5.62) goes to infinity. It does not seem possible to guarantee stability in this regime, unless we require that $\frac{h_{l-1}}{h_l} \leq c_h$, $\forall l$, the way we did it in Theorem 4.1. If enforcing such a condition is inconvenient, the method can be adapted to work by requiring

$$|\Theta^{(i)}(q^{(l)})| \leq Ch_l^2, \quad i = 1, 2, \ldots, m.$$

Therefore, aggressive time step reduction for such a method should be combined with a reduction in the amount of infeasibility in the constraint, by using a few iterations of a nonlinear projection algorithm, to satisfy the above requirement.

6. Numerical Results for Contact Constraints

To validate the concepts introduced in the preceding sections, we applied our method where $v^{(l+1)}$ is computed by (2.10–2.11) to two two-dimensional examples, and we compared it to the unstabilized version (which corresponds to the choice $\Delta = 0$, and $\Upsilon = 0$ in (2.10–2.11). We ran both examples for 20 seconds with a time step of 0.05. The mass data corresponds to a density of 10kg/m^2. All computations were done by solving one linear complementarity problem per step, using PATH [15].

We choose $\widehat{\epsilon}$, the parameter that governs the choice of the active set (4.41), to be equal to 0.3. In the limit of $h_l \to 0$, the value of the active set parameter $\hat{\epsilon}$ is not an issue, as proved in Theorem 4.1. This parameter does influence the efficiency of the algorithm, however, since a larger $\hat{\epsilon}$ means that the size of the LCP (2.10–2.11) will increase. On the other hand, a smaller $\hat{\epsilon}$ means that certain collisions may be missed and could result in a large increase of the infeasibility.

In the first example, we simulate an elliptic body above and on a tabletop. The length of its axes are 8 and 4. The body is dropped from a height of 8 with respect to its center of mass and with an angular velocity of 3. The friction coefficient is 0.3. In Figure 3 we present ten frames of the simulation. In Figure 4 we present a comparison of the constraint infeasibility between the unstabilized and stabilized version of our algorithm. The benefit of the stabilization is evident in the figure where the infeasibility is more than 100 times smaller towards the end of the simulation in the stabilized case compared to the unstabilized case. We also see that in the stabilized case the infeasibility oscillates in a narrow range without exhibiting a substantial increase.

In the second example, we simulate the behavior of 21 identical disks of radius 3 on a horizontal tabletop bounded by two slanted walls, starting from the cannonball arrangement at 0 velocity (with 6 disks at the bottom). The friction coefficient is 0.2. Four frames of the simulation are presented in Figure 5.

In Figure 6 we compare the constraint infeasibility between the unstabilized and the stabilized method. We see that the stabilized method has smaller constraint infeasibility and consistently corrects incidental large infeasibility. At the end of the simulation, all disks are separated, and they are all in contact with the tabletop. The disk on tabletop constraint is satisfied exactly because it is linear in the region of differentiability (4.40), which explains the essentially zero infeasibility in both methods toward the end of the simulation time interval.

In both examples we see that constraint stabilization is achieved by our method, whereas the unstabilized method exhibits a continuous drift in the first example and a larger and more persistent infeasibility in the second example. We also note that in both examples we were able to achieve constraint stabilization by solving only one LCP per step with a constant time step.

7. Conclusions and Future Work

We presented a time-stepping method for rigid multibody dynamics with joints, contact, and friction that provides geometrical constraint stabilization by solving only one LCP per time step. The stabilization is achieved by modifying the right-hand side of the LCP as a function of the infeasibility. In Theorem 4.1 we prove that the velocity sequence stays uniformly bounded over any finite time interval as the time step goes to 0 and that the geometrical constraint infeasibility at step $l+1$ is bounded by a term proportional to $\left|\left|h_l v^{(l+1)}\right|\right|^2$. The fact that the infeasibility is bounded by a local measure of the size of the solution shows that the constraint stabilization is achieved. The results are validated by comparing the stabilized method with the unstabilized method on two examples.

To achieve these results, we have made several assumptions that we plan to relax in future work. In particular, it is important to relax the assumption that the distance functions are differentiable even on a neighborhood of the feasible set, since many simulations need polyhedral bodies. The choice of $\hat{\epsilon}$ also can affect the size of the LCP to be solved, as well as the geometrical constraint infeasibility, and a promising avenue is to choose it as a function of the velocities of the bodies involved and of the time step, while guaranteeing that the energy balance is not destabilized. A question of practical importance is to determine an appropriate size of γ other than 1 (which is the case for which we obtained results in this work) that would result in constraint stabilization while using the rule (2.12) to generate the right-hand side in (2.10). Also of interest to us is to extend these results to the case of elastic or partially elastic collisions.

Finally, an important question is whether the proof of convergence to the solution of a differential inclusion as the time step goes to 0 [30] can be extended to this case. Key facts that are necessary to adapt the proof in [30] have already been proved: that the velocity is bounded and that the LCP (2.10–2.11) is, at most, an $O(h)$ perturbation of the LCP analyzed in [30], both of which follow from Theorem 4.1.

APPENDIX

II. Upper Bounds on Sequences Satisfying Recursive Inequalities

Lemma II.1. *Consider the nonnegative sequences t_n and z_n, for $0 \leq n \leq N$ and h_n for $0 \leq n \leq N-1$, where $t_0 = 0$ and $t_N = T$. Here h_n satisfies $t_{n+1} - t_n = h_n$, for $0 \leq n \leq N-1$,* *and z_n satisfies the inequality*

$$z_{n+1}^2 \leq z_n^2 + h_n c_1(5z_n^2 + 2z_n) + c_2 h_n, \forall 0 \leq n \leq N-1,$$

where $c_1 > 0$ and $c_2 > 0$ are two real parameters. Let $y(t, y_0)$ be the solution of the scalar differential equation

$$\dot{y} = 6c_1 y + (c_1 + c_2)$$

that satisfies $y(0, y_0) = y_0$. Then,

1. *$y(t)$ satisfies*
   (a) *$y(t, x_1) \geq y(t, x_2)$ whenever $x_1 \geq x_2$,*
   (b) *$z_{n+1}^2 \leq y(h_n, z_n^2)$.*
2. *$z_n^2 \leq y(t_n, z_0^2)$, for $0 \leq n \leq N$.*

Proof From the differential equation in $y(t)$, we have that

$$y(t, y_0) = y_0 e^{6c_1 t} + \frac{c_1 + c_2}{6c_1}\left(e^{6c_1 t} - 1\right)$$

which is obviously an increasing function in y_0, which proves Part 1a.

Clearly, $y(t, y_0)$ is an increasing function of t whenever $y_0 \geq 0$. Since $z_n^2 \geq 0$, we can use the fundamental theorem of calculus to obtain that

$$\begin{aligned} y(h_n, z_n^2) - z_n^2 = y(h_n, z_n^2) - y(0, z_n^2) &= h_n \frac{dy}{dt}(\zeta, z_n^2) = h_n\left(6c_1 y(\zeta, z_n^2) + c_1 + c_2\right) \geq \\ & h_n\left(6c_1 y(0, z_n^2) + c_1 + c_2\right) = h_n\left(6c_1 z_n^2 + c_1 + c_2\right), \end{aligned}$$

where $0 \leq \zeta \leq h_n$. Since $2z_n \leq z_n^2 + 1$, we obtain, from the inequality assumed in the statement of this Lemma that z_{n+1}^2 satisfies, that

$$z_{n+1}^2 \leq z_n^2 + 6h_n c_1 z_n^2 + h_n(c_1 + c_2).$$

Comparing this with the inequality involving $y(h_n, z_n^2)$ above, we immediately get that $y(h_n, z_n^2) \geq z_{n+1}^2$, which proves Part 1b.

Finally, we prove Part 2 of the claim by induction. For $n = 0$, it follows from the definition of $y(t, y_0)$. Assume that the claim is proven for $n = k$, or that $y_k \leq y(t_k, z_0^2)$. Using Part 1b, and Part 1a together with the induction hypothesis, we obtain that

$$z_{k+1}^2 \leq y(h_k, z_k^2) \leq y(h_k, y(t_k, z_0^2)) = y(t_k + h_k, z_0^2) = y(t_{k+1}, z_0^2),$$

which proves the inequality for $n = k+1$ and thus the claim. $\diamond$

Theorem II.2. *Consider the nonnegative sequences t_n, z_n, w_n and θ_n for $0 \leq n \leq N$ and h_n for $0 \leq n \leq N-1$, where $\theta_0 = 0$, $t_0 = 0$ and $t_N = T$. Here $h_n > 0$ satisfies $t_{n+1} - t_n = h_n$, for $0 \leq n \leq N-1$. Let $c_i > 0$, $i = 1, 2, \ldots, 5$, and $\psi_1(z, w)$ a continuous mapping of two real arguments that is nonnegative whenever $z \geq 0$ and $w \geq 0$.*

Assume the following:

1. *Whenever*
$$\theta_n \leq c_5, \tag{II.63}$$
for some n satisfying $0 \leq n \leq N-1$, the following inequalities hold:
$$\begin{aligned} z_{n+1}^2 &\leq z_n^2 + h_n c_1 (z_n^2 + w_n z_n + w_n^2 + w_n + z_n) + \frac{c_2}{2} h_n + h_n^2 \psi_1(z_n, w_n) + c_4 \frac{\theta_n^2}{h_n^2} \quad \text{(II.64)} \\ w_{n+1} &\leq w_n + c_1 h_n z_{n+1}. \quad \text{(II.65)} \end{aligned}$$
2. *If, in addition,*
$$c_1 h_n z_{n+1} \leq c_5, \tag{II.66}$$
then the following inequality also holds for $0 \leq n \leq N-1$:
$$\theta_{n+1} \leq c_3 h_n^2 z_{n+1}^2. \tag{II.67}$$
3. *The time steps h_n, $n = 1, 2, \ldots, N-1$ are chosen such that*
$$\frac{h_{n-1}}{h_n} \leq c_h, \tag{II.68}$$
where $c_h > 0$ is a fixed parameter.

Then there exists an $H > 0$ such that, whenever $h_n < H$, $\forall 0 \leq n \leq N-1$, we have that (II.63), (II.66) and thus (II.64), (II.65) and (II.67) hold for any $0 \leq n \leq N-1$ and that $z_n^2 \leq y(t_n, \max\{z_0, w_0\}^2)$ and $w_n^2 \leq y(t_n, \max\{z_0, w_0\}^2)$, $\forall 0 < n < N$. Here $y(t, y_0)$ is the function defined in Lemma II.1.

Proof

Assume that (II.63) holds and, from the first assumption, that (II.65) holds. Using the fact that all relevant sequences are nonnegative and taking the square of both sides, we obtain that
$$w_{n+1}^2 \leq w_n^2 + 2h_n c_1 z_{n+1} w_n + h_n^2 c_1^2 z_{n+1}^2 \leq w_n^2 + h_n c_1 z_{n+1}^2 + h_n c_1 w_n^2 + h_n^2 c_1^2 z_{n+1}^2.$$
Define
$$q_n = \max\{z_n, w_n\}, \tag{II.69}$$
for $n = 0, 1, \ldots, N-1$. Using this in (II.64) and the previous inequality, we obtain that whenever (II.63) holds, we have that
$$\begin{aligned} z_{n+1}^2 &\leq q_n^2 + c_1 h_n (3q_n^2 + 2q_n) + \frac{c_2}{2} h_n + h_n^2 \psi_1(z_n, w_n) + c_4 \frac{\theta_n^2}{h_n^2} \\ w_{n+1}^2 &\leq q_n^2 + h_n c_1 q_{n+1}^2 + h_n c_1 q_n^2 + h_n^2 c_1^2 q_{n+1}^2. \end{aligned}$$
Since all terms in these inequalities are nonnegative, using that $q_n^2 \leq 3q_n^2 + 2q_n$ and that $q_{n+1}^2 \leq \max\{w_n^2, z_n^2\}$, we obtain that
$$q_{n+1}^2 \leq q_n^2 + c_1 h_n q_{n+1}^2 + c_1^2 h_n^2 q_{n+1}^2 + c_1 h_n (3q_n^2 + 2q_n) + \frac{c_2}{2} h_n + h_n^2 \psi_1(z_n, w_n) + c_4 \frac{\theta_n^2}{h_n^2}.$$
Here we have used that $z_{n+1}^2 \leq a + b_1 + c_1$ and $w_{n+1}^2 \leq a + b_2 + c_2$, where $b_1 \geq b_2 \geq 0$ and $a, c_1, c_2 \geq 0$ implies that $q_{n+1}^2 \leq a + b_1 + c_1 + c_2$. Using now in the previous displayed inequality that
$$\frac{1}{1 - a - a^2} \leq 1 + 2a, \quad \forall a \in \left[0, \frac{1}{4}\right],$$

we obtain that as soon as

$$c_1 h_n \leq \frac{1}{4} \text{ and } \theta_n \leq c_5,$$

we have that

$$\begin{aligned} q_{n+1}^2 &\leq (1+2c_1h_n)\left[q_n^2 + c_1h_n(3q_n^2+2q_n) + \frac{c_2}{2}h_n + h_n^2\psi_1(z_n,w_n) + c_4\frac{\theta_n^2}{h_n^2}\right] \\ &\leq q_n^2 + c_1h_n\left(5q_n^2+2q_n\right) + \frac{c_2}{2}h_n + \frac{3}{2}c_4\frac{\theta_n^2}{h_n^2} + h_n^2\widehat{\psi}_1(z_n,w_n), \end{aligned} \tag{II.70}$$

where

$$\widehat{\psi}_1(z_n,w_n) = c_1c_2 + 2c_1^2(3q_n^2+2q_n) + \frac{3}{2}\psi_1(z_n,w_n).$$

Here we used that, whenever we have that $c_1 h_n \leq \frac{1}{4}$ we also have that $1 + 2c_1 h_n \leq \frac{3}{2}$. If, in addition, we have that $h_n c_1 z_{n+1} \leq c_5$, then from Assumption 2 we have that

$$\theta_{n+1} \leq c_3 h_n^2 z_{n+1}^2 \leq c_3 h_n^2 q_{n+1}^2. \tag{II.71}$$

In the following we use $y(t,q)$, function defined in Lemma II.1, where we use as c_1 and c_2 the parameters c_1 and c_2 from this theorem.

Choose now H to be the largest h that satisfies the following inequalities:

$$\frac{c_2}{4} \geq h \max_{0\leq v,w\leq\sqrt{y(T,q_0)}} \widehat{\psi}_1(v,w) \tag{II.72}$$

$$\frac{c_2}{4} \geq hc_4c_3^2c_h^4\frac{3}{2}y(T,q_0)^2 \tag{II.73}$$

$$c_5 \geq h^2c_3y(T,q_0) \tag{II.74}$$

$$\frac{1}{4} \geq hc_1 \tag{II.75}$$

$$c_5 \geq hc_1\sqrt{y(T,q_0)}. \tag{II.76}$$

We now prove by induction that, with this choice of H, we will have that

$$\theta_n \leq c_5 \tag{II.77}$$

$$q_n^2 \leq y(t_n,q_0^2) \leq y(T,q_0^2) \tag{II.78}$$

for all $0 \leq n \leq N$, whenever

(H1) $h_n \leq H$ for $0 \leq n \leq N$ and
(H2) $\frac{h_{n-1}}{h_n} \leq c_h$ for $n = 0,1,2,\ldots,N-1$.

Case $n=0$*:* From our initial assumptions, we have that $\theta_0 = 0$, which satisfies (II.77). Using Lemma II.1, we have that $q_0^2 = y(0,q_0^2) \leq y(T,q_0^2)$, and thus (II.78) is satisfied.

Case $n=n_a$*:* We now assume that (II.77) and (II.78) hold for $n = 1,2,\ldots,n_a$, and we intend to prove that they also hold for n_a+1.

Since (II.78) holds for $n = 1,2,\ldots,n_a$, using the property (H1) of the time step sequence, the definition of q_n (II.69), as well as (II.72), we obtain that

$$h_n\widehat{\psi}(z_n,w_n) \leq \frac{c_2}{4},\ n = 0,1,2,\ldots n_a. \tag{II.79}$$

Also, since (II.78) applies for $n = 0, 1, 2, \ldots, n_a$, we must have that

$$c_1 h_{n-1} z_n \leq c_1 h_{n-1} q_n \leq c_1 h_{n-1} \sqrt{y(T, q_0^2)} \leq c_5$$

for $n = 1, 2, \ldots, n_a$, where the last inequality follows from the assumption (H1) as well as (II.76). By the induction hypothesis, we have that $\theta_n \leq c_5$, for $n = 0, 1, 2 \ldots, n_a$. The last two inequalities ensure then that (II.63) and (II.66), and thus (II.64) and (II.65), hold for $n = 1, 2, \ldots, n_a$, and that (II.67) holds for $n = 0, 1, 2, \ldots, n_a - 1$.

Since (II.67) holds for $n = 1, 2, \ldots, n_a - 1$, we must have that $\theta_n \leq c_3 h_{n-1}^2 q_n^2$ for $n = 1, 2, \ldots, n_a$. We thus obtain, using the assumption (H2), that

$$\frac{\theta_n^2}{h_n^2} = \frac{\theta_n^2}{h_{n-1}^2} \frac{h_{n-1}^2}{h_n^2} \leq c_3^2 h_{n-1}^2 q_n^4 \frac{h_{n-1}^2}{h_n^2} = c_3^2 h_n^2 q_n^4 \frac{h_{n-1}^4}{h_n^4} \leq c_3^2 h_n^2 q_n^4 c_h^4,$$

for $n = 1, 2, \ldots, n_a$. Using (II.73) and the choice (H1) of H, we obtain that

$$c_4 \frac{3}{2} \frac{\theta_n^2}{h_n^2} \leq \frac{c_2}{4} h_n, \quad n = 1, 2, \ldots, n_a.$$

Using the last inequality together with (II.79) in (II.70), which holds for $n = n_a$ because (II.64) and (II.65) hold, we obtain that

$$q_{n+1}^2 \leq q_n^2 + c_1 h_n (5 q_n^2 + 2 q_n) + c_2 h_n, \quad n = 0, 1, 2, \ldots, n_a.$$

Using Lemma II.1, we have that

$$q_{n_a+1} \leq y(t_{n_a+1}, q_0^2) \leq y(T, q_0^2),$$

which prove (II.78) for $n = n_a + 1$. Using (H1) and (II.76), we have that $c_1 h_{n_a} q_{n_a+1} \leq c_5$. Therefore (II.71) applies, to give us

$$\theta_{n_a+1} \leq c_3 h_{n_a}^2 q_{n_a+1}^2 \leq c_5,$$

where the last inequality follows by using (II.78) for $n = n_a + 1$ and (II.74). This shows that (II.77) holds for $n = n_a + 1$, which completes our induction proof, and thus (II.77) and (II.78) hold for $n = 0, 1, 2, \ldots, N$.

Since we have that $q_n = \max\{w_n, z_n\}$, and therefore that $c_1 h_n z_{n+1} \leq c_1 h_n q_{n+1} \leq c_3$ (the last inequality following from (II.74) and (H1)), we obtain that (II.63), (II.66) and (II.78) hold for $n = 0, 1, 2, \ldots, N$, which completes the proof, after applying Assumptions 1 and 2 of this theorem. ◇

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REFERENCES

1. Anitescu, M., and Hart, Gary D. "A fixed-point iteration approach for multibody dynamics with contact and small friction", Preprint ANL/MCS-P985-0802, Mathematics and Computer Science Division, Argonne National Laboratory, 2002.
2. Anitescu, M., and Potra, F. A., "A time-stepping method for stiff multibody dynamics with contact and friction", *International Journal for Numerical Methods in Engineering* **55**(7), 753–784, 2002.
3. Anitescu, M., and Potra, F. A., "Formulating rigid multi-body-dynamics with contact and friction as solvable linear complementarity problems", *Nonlinear Dynamics* **14**, 231–247, 1997.
4. Anitescu, M., Stewart, D., and Potra,F. A., "Time-stepping for three-dimensional rigid body dynamics", *Computer Methods in Applied Mechanics and Engineering* **177**(3–4), 183–197, 1999.
5. Anitescu, M., Cremer, J., and Potra, F. A., "Formulating 3D contact dynamics problems", *J. Mech. Struct. Mach.* **24**(4), 405–437, 1996.
6. Anitescu, M., Cremer, J. F., and Potra, F. A. "Properties of complementarity formulations for contact problems with friction", in *Complementarity and Variational Problems: State of the Art*, edited by Michael C. Ferris and Jong-Shi Pang, SIAM, Philadelphia, 1997, pp. 12–21.
7. Ascher, U., Chin. H. and Reich, S. "Stabilization of DAEs and invariant manifolds", *Numerische Mathematik*, **67**, 131–149, 1994.
8. Ascher, U. and Petzold, L. *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*, SIAM, Philadelphia, 1998.
9. Ascher, U., Chin, H. Petzold, L., and Reich, S. "Stabilization of constrained mechanical systems with DAEs and invariant manifolds", *J. Mech. Struct. Mach*, **23**, 135-158, 1995.
10. Baumgarte, J., "Stabilization of constraints and integrals of motion in dynamical systems", *Computer Methods in Applied Mechanics and Engineering*, **1**, 1–16, 1972.
11. Baraff, D., " Issues in computing contact forces for non-penetrating rigid bodies", *Algorithmica* **10**, 292–352, 1993.
12. Cline, M., *Rigid Body Simulation with Contact and Constraints*, M.S. thesis, Department of Computer Science, University of British Columbia, 2002.
13. Miller, A., and Christensen H. I., "Implementation of multi-rigid-body dynamics within a robotics grasping simulator", preprint, 2002. Submitted to the IEEE International Conference on Robotics and Automation.
14. Cremer, J. F., and Stewart, A. J. " The architecture of Newton, a general purpose dynamics simulator", *Proceeding of the IEEE International Conference on Robotics and Automation*, 1989, pp. 1806–1811. Int'l IEEE
15. Dirkse, S. P., and Ferris, M. C., "The PATH solver: A non-monotone stabilization scheme for mixed complementarity problems", *Optimization Methods and Software* **5**, 123–156, 1995.
16. Donald, B. R., and Pai, D. K. "On the motion of compliantly connected rigid bodies in contact: a system for analyzing designs for assembly", *Proceedings of the Conf. on Robotics and Automation*, 1990, pp. 1756–1762.
17. Lo, G., Sudarsky, S., Pang, J.-S., and Trinkle, J., "On dynamic multi-rigid-body contact problems with Coulomb friction," *Zeitschrift fur Angewandte Mathematik und Mechanik* **77**, 267–279, 1997.
18. Kim, Y. J., Lin, M. C., and Manocha, D., "DEEP: Dual-space expansion for estimating penetration depth between convex polytopes" *Proceedings of the IEEE International Conference on Robotics and Automation,* 2002.
19. Mirtich, B., *Impulse-based Dynamic Simulation of Rigid Body Systems*, Ph.D. thesis, Department of Computer Science, University of California, Berkeley, December, 1996.
20. Haug, E. J., Wu, S. C. and Yang, S. M., "Dynamic mechanical systems with Coulomb friction, stiction, impact and constraint addition-deletion I: Theory", *Mechanisms and Machine Theory* **21**(5), 407–416, 1986.
21. Haug, E. J., *Computer Aided Kinematics and Dynamics of Mechanical Systems*, Allyn and Bacon, Boston, 1989.
22. Mangasarian, O. L., *Nonlinear Programming*, McGraw-Hill, New York 1969.
23. Mangasarian, O. L. and Fromovitz, S., "The Fritz John necessary optimality conditions in the presence of equality constraints", *Journal of Mathematical Analysis and Applications* **17**, 34–47, 1967.
24. Munson, T. S., *Algorithms and Environments for Complementarity*, Ph.D. thesis, Department of Computer Science, University of Wisconsin-Madison, 2000.
25. Murray R. M., Li, Z., and Sastry, S. S., *Robotic Manipulation*, CRC Press, Boca Raton, FL, 1993.
26. Pang, J.-S., and Stewart, D.E., "A unified approach to frictional contact problems", *International Journal of Engineering Science*, **37**, 1747–1768, 1999.
27. Robinson, S.M., "Generalized equations and their solutions, part II: Applications to nonlinear programming", *Mathematical Progamming Study* **19**, 200–221, 1982.
28. P. Song, P. Kraus, V. Kumar, and P. Dupont, "Analysis of Rigid-Body Dynamic Models for Simulation of Systems With Frictional Contacts" *Journal of Applied Mechanics* **68(1)**, 118-128, 2001.
29. Stewart, D. E., and Trinkle, J. C., "An implicit time-stepping scheme for rigid-body dynamics with inelastic

collisions and Coulomb friction", *International J. Numerical Methods in Engineering* **39**, 2673-2691, 1996.
30. Stewart, D., "Rigid-body dynamics with friction and impact", *SIAM Review* **42** (1), 3–29, 2000.

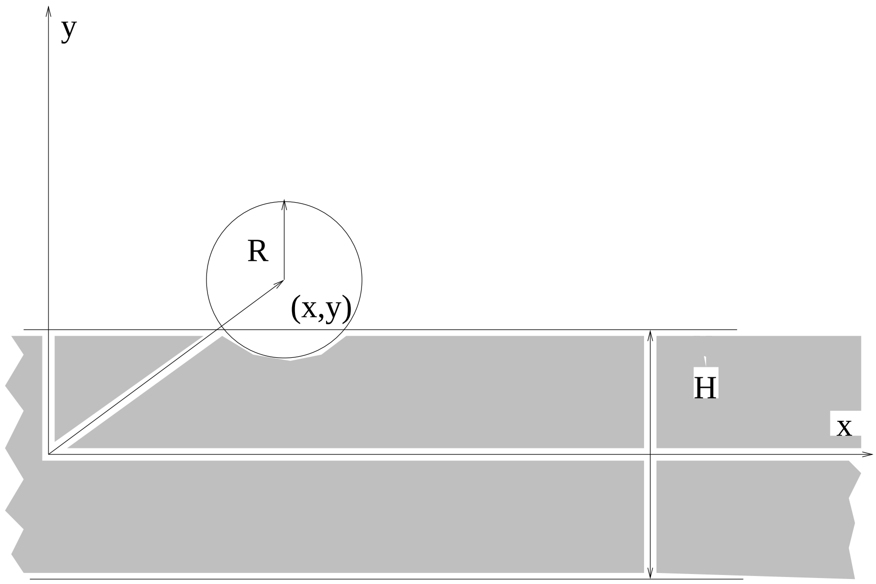

Figure 1. Two-body interpenetration configuration

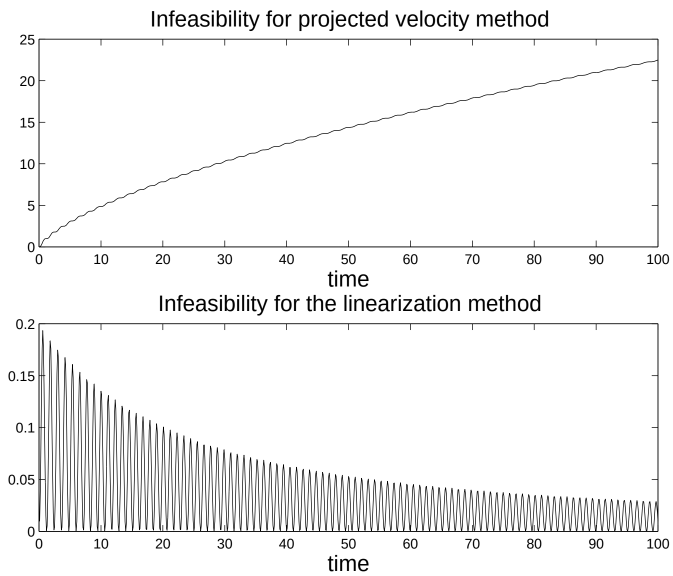

Figure 2. Comparison between our (linearization) method) and the unstabilized (velocity projection) method.

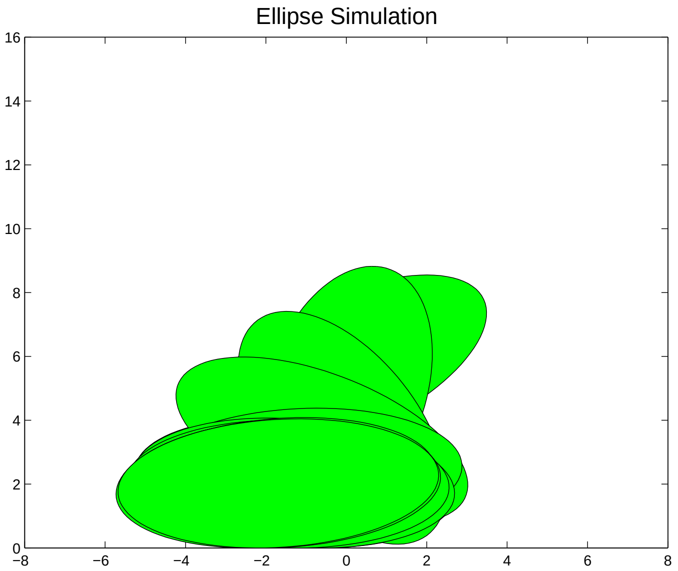

Figure 3. Ten frames of an ellipse on a tabletop simulation.

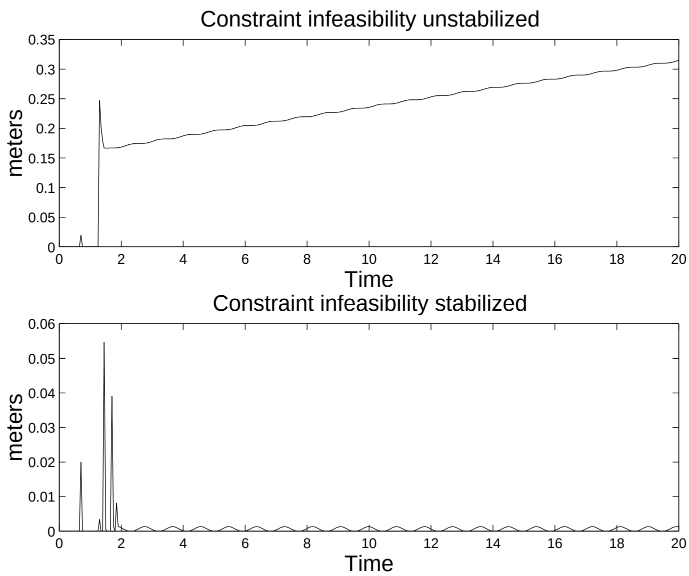

Figure 4. Ellipse simulation: Comparison of the constraint infeasibility between the unstabilized method and the stabilized method.

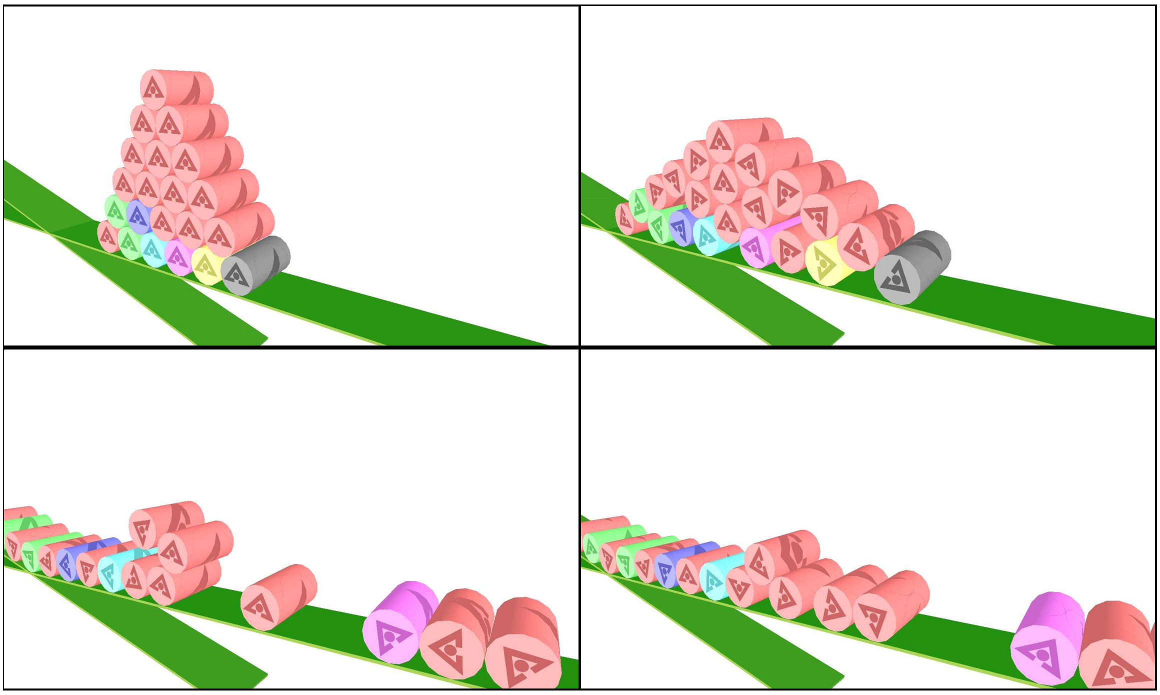

Figure 5. Four frames of a two-dimensional cannonball arrangement simulation involving 21 bodies

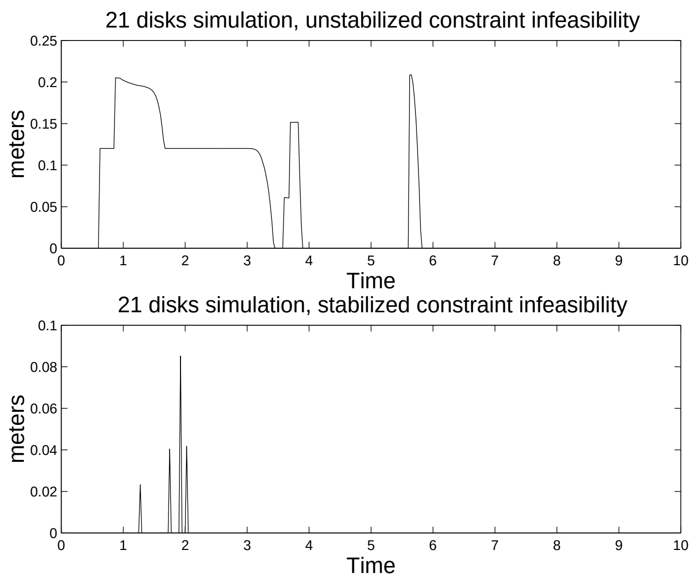

Figure 6. Disks simulation: Comparison of the constraint infeasibility between the unstabilized method and the stabilized method.